



**Faculty of Engineering**  
**Mechanical Engineering Department**

# **CALCULUS FOR ENGINEERS**

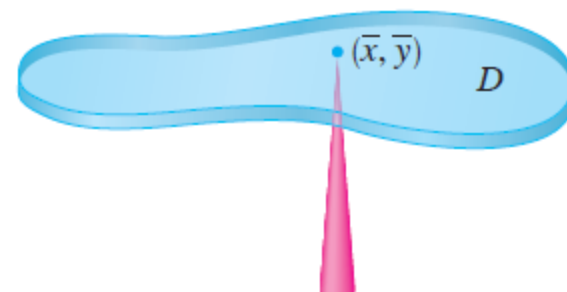
## **MATH 1110**

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# Moments and Centers of Mass

$$M_x = \iint_D y \rho(x, y) \, dA$$

$$M_y = \iint_D x \rho(x, y) \, dA$$



The coordinates  $(\bar{x}, \bar{y})$  of the center of mass of a lamina occupying the region  $D$  and having density function  $\rho(x, y)$  are

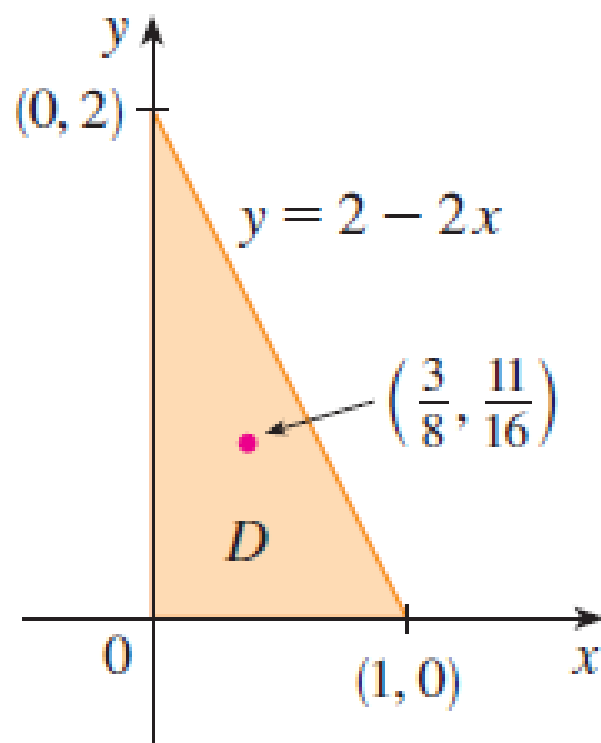
$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \rho(x, y) \, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \rho(x, y) \, dA$$

where the mass  $m$  is given by

$$m = \iint_D \rho(x, y) \, dA$$

# Example 1

Find the mass and center of mass of a triangular lamina with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 2)$  if the density function is  $\rho(x, y) = 1 + 3x + y$ .



## SOLUTION

$$m = \iint_D \rho(x, y) \, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y) \, dy \, dx$$

$$= \int_0^1 \left[ y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2) \, dx = 4 \left[ x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) \, dA = \frac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy) \, dy \, dx$$

$$= \frac{3}{8} \int_0^1 \left[ xy + 3x^2 y + x \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

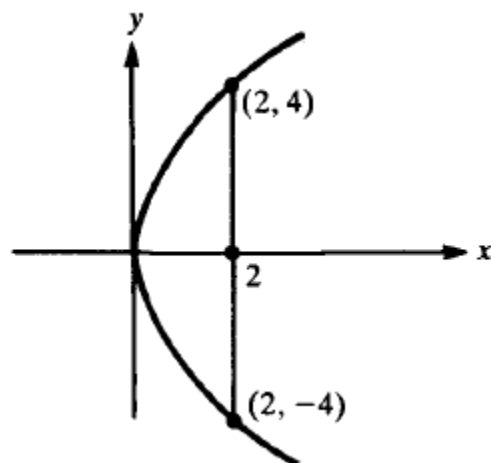
$$= \frac{3}{2} \int_0^1 (x - x^3) \, dx = \frac{3}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

$$\begin{aligned}
\bar{y} &= \frac{1}{m} \iint_D y \rho(x, y) \, dA = \frac{3}{8} \int_0^1 \int_0^{2-2x} (y + 3xy + y^2) \, dy \, dx \\
&= \frac{3}{8} \int_0^1 \left[ \frac{y^2}{2} + 3x \frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx = \frac{1}{4} \int_0^1 (7 - 9x - 3x^2 + 5x^3) \, dx \\
&= \frac{1}{4} \left[ 7x - 9 \frac{x^2}{2} - x^3 + 5 \frac{x^4}{4} \right]_0^1 = \frac{11}{16}
\end{aligned}$$

The center of mass is at the point  $(\frac{3}{8}, \frac{11}{16})$ .

# Example 2

Find the center of mass  $(\bar{x}, \bar{y})$  of the plate cut from the parabola  $y^2 = 8x$  by its latus rectum  $x = 2$  if the density is numerically equal to the distance from the latus rectum.



# SOLUTION

By symmetry,  $\bar{y} = 0$ .

The mass  $M = \iint (2-x) dA = \int_{-4}^4 \int_{y^2/8}^2 (2-x) dx dy =$

$$\int_{-4}^4 \left( 2x - \frac{1}{2} x^2 \right) \Big|_{y^2/8}^2 dy = \int_{-4}^4 \left[ 2 - \left( \frac{y^2}{4} - \frac{y^4}{128} \right) \right] dy =$$

$$\left( 2y - \frac{1}{12} y^3 + \frac{1}{128} \frac{y^5}{5} \right) \Big|_{-4}^4 = 8 \left( 2 - \frac{16}{12} + \frac{1}{128} \cdot \frac{256}{5} \right) =$$

$$\frac{128}{15}$$

The moment about the  $y$ -axis is given by

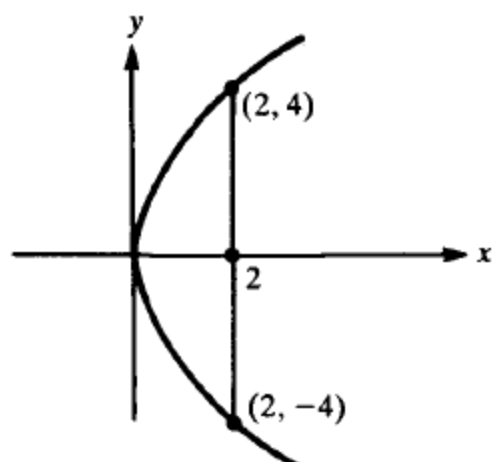
$$M_y = \int_{-4}^4 \int_{y^2/8}^2 x(2-x) dx dy$$

$$= \int_{-4}^4 \left( x^2 - \frac{1}{3} x^3 \right) \Big|_{y^2/8}^2 dy =$$

$$\int_{-4}^4 \left[ \frac{4}{3} \left( \frac{y^4}{64} - \frac{y^6}{3 \cdot 512} \right) \right] dy = \left( \frac{4y}{3} - \frac{1}{64} \frac{y^5}{5} + \frac{1}{3 \cdot 512} \frac{y^7}{7} \right) \Big|_{-4}^4 =$$

$$8 \left( \frac{4}{3} - \frac{4}{5} + \frac{8}{21} \right) = \frac{256}{35}. \quad \text{Hence, } \bar{x} = \frac{M_y}{M} =$$

$$\frac{\frac{256}{35}}{\frac{128}{15}} = \frac{6}{7}. \quad \text{Thus, the center of mass is } \left( \frac{6}{7}, 0 \right).$$



## Example 3

The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

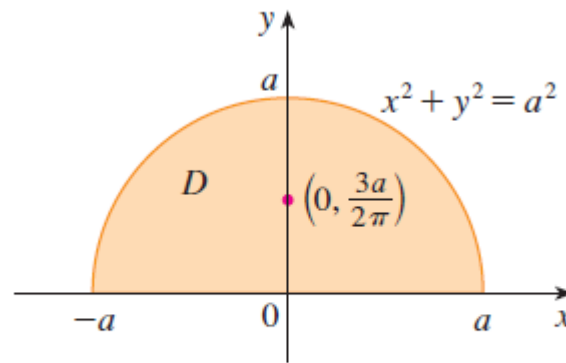


FIGURE 6



**SOLUTION** Let's place the lamina as the upper half of the circle  $x^2 + y^2 = a^2$ . (See Figure 6.) Then the distance from a point  $(x, y)$  to the center of the circle (the origin) is  $\sqrt{x^2 + y^2}$ . Therefore the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where  $K$  is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then  $\sqrt{x^2 + y^2} = r$  and the region  $D$  is given by  $0 \leq r \leq a$ ,  $0 \leq \theta \leq \pi$ . Thus the mass of the lamina is

$$\begin{aligned} m &= \iint_D \rho(x, y) \, dA = \iint_D K\sqrt{x^2 + y^2} \, dA \\ &= \int_0^\pi \int_0^a (Kr) \, r \, dr \, d\theta = K \int_0^\pi d\theta \int_0^a r^2 \, dr \\ &= K\pi \left[ \frac{r^3}{3} \right]_0^a = \frac{K\pi a^3}{3} \end{aligned}$$

Both the lamina and the density function are symmetric with respect to the  $y$ -axis, so the center of mass must lie on the  $y$ -axis, that is,  $\bar{x} = 0$ . The  $y$ -coordinate is given by

$$\begin{aligned}\bar{y} &= \frac{1}{m} \iint_D y \rho(x, y) \, dA = \frac{3}{K\pi a^3} \int_0^\pi \int_0^a r \sin \theta (Kr) \, r \, dr \, d\theta \\ &= \frac{3}{\pi a^3} \int_0^\pi \sin \theta \, d\theta \int_0^a r^3 \, dr = \frac{3}{\pi a^3} [-\cos \theta]_0^\pi \left[ \frac{r^4}{4} \right]_0^a \\ &= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}\end{aligned}$$

Therefore the center of mass is located at the point  $(0, 3a/(2\pi))$ .