## Revision problems from tutorial 12 to 15

## Iterated integrals

**1–2** Find $\int_0^3 f(x, y)\, dx$ and $\int_0^4 f(x, y)\, dy$.

**1.** $f(x, y) = 2x + 3x^2y$

**3–12** Calculate the iterated integral.

**3.** $\int_1^3 \int_0^1 (1 + 4xy)\, dx\, dy$

**4.** $\int_2^4 \int_{-1}^1 (x^2 + y^2)\, dy\, dx$

**6.** $\int_1^4 \int_0^2 (x + \sqrt{y})\, dx\, dy$

**7.** $\int_0^2 \int_0^1 (2x + y)^8\, dx\, dy$

**8.** $\int_0^1 \int_1^2 \frac{xe^x}{y}\, dy\, dx$

**9.** $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx$

**10.** $\int_1^2 \int_0^1 (x + y)^{-2}\, dx\, dy$

11. $\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,dx\,dy$

12. $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx$

13–20 Calculate the double integral.

13. $\iint_R (6x^2y^3 - 5y^4)\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 1\}$

14. $\iint_R \cos(x + 2y)\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi/2\}$

15. $\iint_R \frac{xy^2}{x^2+1}\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ -3 \leqslant y \leqslant 3\}$

16. $\displaystyle\iint_R \frac{1+x^2}{1+y^2}\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$

17. $\displaystyle\iint_R x\sin(x+y)\,dA, \quad R = [0, \pi/6] \times [0, \pi/3]$

18. $\displaystyle\iint_R \frac{x}{1+xy}\,dA, \quad R = [0, 1] \times [0, 1]$

19. $\displaystyle\iint_R xye^{x^2 y}\,dA, \quad R = [0, 1] \times [0, 2]$

20. $\displaystyle\iint_R \frac{x}{x^2+y^2}\,dA, \quad R = [1, 2] \times [0, 1]$

1. $\int_0^3 \left(2x+3x^2y\right)dx=\left[x^2+x^3y\right]_{x=0}^{x=3}=(9+27y)-(0+0)=9+27y$ ,

$$\int_0^4 \left(2x+3x^2y\right)dy=\left[\,2xy+3x^2\,\frac{y^2}{2}\,\right]_{y=0}^{y=4}=\left(8x+3x^2\cdot\frac{16}{2}\right)-(0+0)=8x+24x^2$$

3. $\int_1^3\int_0^1 (1+4xy)\,dx\,dy=\int_1^3\left[x+2x^2y\right]_{x=0}^{x=1}dy=\int_1^3(1+2y)\,dy=\left[y+y^2\right]_1^3=(3+9)-(1+1)=10$

4.

$$\int_2^4\int_{-1}^1 \left(x^2+y^2\right)dy\,dx=\int_2^4\left[\,x^2y+\frac{1}{3}\,y^3\,\right]_{y=-1}^{y=1}dx=\int_2^4\left[\left(x^2+\frac{1}{3}\right)-\left(-x^2-\frac{1}{3}\right)\right]dx$$

$$=\int_2^4\left(2x^2+\frac{2}{3}\right)dx=\left[\,\frac{2}{3}\,x^3+\frac{2}{3}\,x\,\right]_2^4=\left(\frac{128}{3}+\frac{8}{3}\right)-\left(\frac{16}{3}+\frac{4}{3}\right)=\frac{116}{3}$$

6.

$$\int_1^4\int_0^2 (x+\sqrt{y})\,dx\,dy = \int_1^4 \left[\frac{1}{2}x^2 + x\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 (2+2\sqrt{y})\,dy$$

$$= \left[2y + 2\cdot\frac{2}{3}y^{3/2}\right]_1^4 = \left(8+\frac{4}{3}\cdot 8\right) - \left(2+\frac{4}{3}\right) = \frac{46}{3}$$

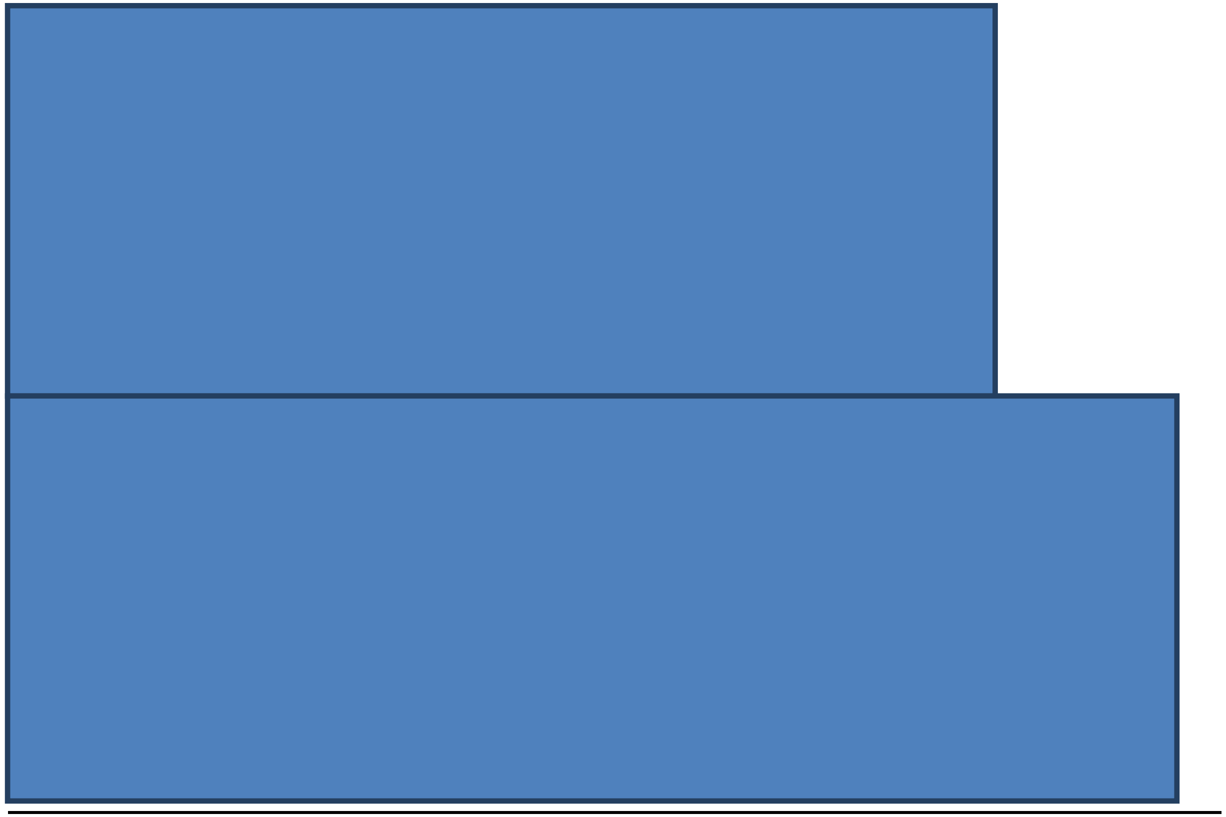
7.

$$\int_0^2\int_0^1 (2x+y)^8\,dx\,dy = \int_0^2 \left[\frac{1}{2}\,\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1} dy$$

Let u = 2x+y

$$= \frac{1}{18}\int_0^2 [(2+y)^9-(0+y)^9]\,dy = \frac{1}{18}\left[\frac{(2+y)^{10}}{10}-\frac{y^{10}}{10}\right]_0^2$$

$$= \frac{1}{180}[(4^{10}-2^{10})-(2^{10}-0^{10})] = \frac{1,046,528}{180} = \frac{261,632}{45}$$



10.

$$\int_1^2 \int_0^1 (x+y)^{-2} dx\, dy = \int_1^2 \left[ -(x+y)^{-1} \right]_{x=0}^{x=1} dy = \int_1^2 \left[ y^{-1} - (1+y)^{-1} \right] dy$$

$$= \left[ \ln y - \ln (1+y) \right]_1^2 = \ln 2 - \ln 3 - 0 + \ln 2 = \ln \frac{4}{3}$$

12.

$$\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1\left[x\sqrt{x^2+y^2+1}\right]_{y=0}^{y=1}dx = \int_0^1 x\left(\sqrt{x^2+2}-\sqrt{x^2+1}\right)dx$$

$$\boxed{U = X^2+y^2+1}$$

$$= \frac{1}{3}\left[(x^2+2)^{3/2}-(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}\left[(3^{3/2}-2^{3/2})-(2^{3/2}-1)\right]$$

$$= \frac{1}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right)$$

13.

$$\iint_R (6x^2y^3-5y^4)\,dA = \int_0^3\int_0^1 (6x^2y^3-5y^4)\,dy\,dx = \int_0^3\left[\frac{3}{2}x^2y^4-y^5\right]_{y=0}^{y=1}dx$$

$$= \int_0^3\left(\frac{3}{2}x^2-1\right)dx = \left[\frac{1}{2}x^3-x\right]_0^3 = \frac{27}{2}-3 = \frac{21}{2}$$

---

14.

$$\iint_R \cos(x+2y)\,dA = \int_0^{\pi}\int_0^{\pi/2} \cos(x+2y)\,dy\,dx$$

U= x+2y

$$= \int_0^{\pi}\left[\frac{1}{2}\sin(x+2y)\right]_{y=0}^{y=\pi/2} dx = \frac{1}{2}\int_0^{\pi}(\sin(x+\pi)-\sin x)\,dx$$

$$= \frac{1}{2}\left[-\cos(x+\pi)+\cos x\right]_0^{\pi} = \frac{1}{2}\left[-\cos 2\pi+\cos\pi-(-\cos\pi+\cos 0)\right]$$

$$= \frac{1}{2}(-1-1-(1+1)) = -2$$

15.

$$\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\,dx \int_{-3}^3 y^2\,dy$$

$$= \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 \left[\frac{1}{3}y^3\right]_{-3}^3 = \frac{1}{2}(\ln 2-\ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$$

## Triple integrals

**2.** Evaluate the integral $\iiint_E (xz - y^3)\, dV$, where

$$E = \{(x, y, z) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2, 0 \leqslant z \leqslant 1\}$$

using three different orders of integration.

**3–6** Evaluate the iterated integral.

**3.** $\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$

**4.** $\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx$

**5.** $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y\, dx\, dz\, dy$

**6.** $\int_0^1 \int_0^z \int_0^y ze^{-y^2}\, dx\, dy\, dz$

**7.** $\iiint_E 2x\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant \sqrt{4 - y^2},\ 0 \leqslant z \leqslant y\}$

**8.** $\iiint_E yz\cos(x^5)\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x,\ x \leqslant z \leqslant 2x\}$

**9.** $\iiint_E 6xy\, dV$, where $E$ lies under the plane $z = 1 + x + y$ and above the region in the $xy$-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

**10.** $\iiint_E y\, dV$, where $E$ is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$

**11.** $\iiint_E xy\, dV$, where $E$ is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$

**12.** $\iiint_E xz\, dV$, where $E$ is the solid tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$

**13.** $\iiint_E x^2 e^y\, dV$, where $E$ is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$

2. There are six different possible orders of integration.

$$\iiint_E (xz-y^3)\,dV=\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(xz-y^3)\,dz\,dy\,dx=\int_{-1}^{1}\int_{0}^{2}\left[\frac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dy\,dx$$

$$=\int_{-1}^{1}\int_{0}^{2}\left(\frac{1}{2}x-y^3\right)dy\,dx=\int_{-1}^{1}\left[\frac{1}{2}xy-\frac{1}{4}y^4\right]_{y=0}^{y=2}dx$$

$$=\int_{-1}^{1}(x-4)\,dx=\left[\frac{1}{2}x^2-4x\right]_{-1}^{1}=-8$$

$$\iiint_E (xz-y^3)\,dV=\int_{0}^{2}\int_{-1}^{1}\int_{0}^{1}(xz-y^3)\,dz\,dx\,dy=\int_{0}^{2}\int_{-1}^{1}\left[\frac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dx\,dy$$

$$=\int_{0}^{2}\int_{-1}^{1}\left(\frac{1}{2}x-y^3\right)dx\,dy=\int_{0}^{2}\left[\frac{1}{4}x^2-xy^3\right]_{x=-1}^{x=1}dy$$

$$=\int_{0}^{2}-2y^3\,dy=\left.-\frac{1}{2}y^4\right]_{0}^{2}=-8$$

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^{1}\int_{0}^{1}\int_{0}^{2} (xz-y^3)\,dy\,dz\,dx = \int_{-1}^{1}\int_{0}^{1}\left[xyz-\frac{1}{4}y^4\right]_{y=0}^{y=2} dz\,dx$$

$$= \int_{-1}^{1}\int_{0}^{1} (2xz-4)\,dz\,dx = \int_{-1}^{1}\left[xz^2-4z\right]_{z=0}^{z=1} dx$$

$$= \int_{-1}^{1} (x-4)\,dx = \left[\frac{1}{2}x^2-4x\right]_{-1}^{1} = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_{0}^{1}\int_{-1}^{1}\int_{0}^{2} (xz-y^3)\,dy\,dx\,dz = \int_{0}^{1}\int_{-1}^{1}\left[xyz-\frac{1}{4}y^4\right]_{y=0}^{y=2} dx\,dz$$

$$=\int_0^1\int_{-1}^1 (2xz-4)\,dx\,dz=\int_0^1 \left[x^2z-4x\right]_{x=-1}^{x=1} dz$$

$$=\int_0^1 -8\,dz= -8z\Big]_0^1=-8$$

$$\iiint_E (xz-y^3)\,dV=\int_0^2\int_0^1\int_{-1}^1 (xz-y^3)\,dx\,dz\,dy=\int_0^2\int_0^1 \left[\frac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1} dz\,dy$$

$$=\int_0^2\int_0^1 -2y^3\,dz\,dy=\int_0^2 \left[-2y^3z\right]_{z=0}^{z=1} dy=\int_0^2 -2y^3\,dy= -\frac{1}{2}y^4\Big]_0^2=-8$$

$$\iiint_E \left(xz-y^3\right)dV=\int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\,dx\,dy\,dz=\int_0^1\int_0^2 \left[\frac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1} dy\,dz$$

$$=\int_0^1\int_0^2 -2y^3\,dy\,dz=\int_0^1 \left[-\frac{1}{2}y^4\right]_{y=0}^{y=2} dz=\int_0^1 -8\,dz= -8z\Big]_0^1=-8$$

3.

$$\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1\int_0^z [6xyz]_{y=0}^{y=x+z} dx\,dz = \int_0^1\int_0^z 6xz(x+z)\,dx\,dz$$

$$= \int_0^1 \left[2x^3z + 3x^2z^2\right]_{x=0}^{x=z} dz = \int_0^1 (2z^4+3z^4)\,dz = \int_0^1 5z^4\,dz = \left. z^5\right]_0^1 = 1$$

4.

$$\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx = \int_0^1\int_x^{2x} \left[xyz^2\right]_{z=0}^{z=y} dy\,dx = \int_0^1\int_x^{2x} xy^3\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x} dx = \int_0^1 \frac{15}{4}x^5\,dx = \left.\frac{5}{8}x^6\right]_0^1 = \frac{5}{8}$$

5.

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy=\int_0^3\int_0^1\left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}}dz\,dy=\int_0^3\int_0^1 ze^y\sqrt{1-z^2}\,dz\,dy$$

$$=\int_0^3\left[-\frac{1}{3}(1-z^2)^{3/2}e^y\right]_{z=0}^{z=1}dy=\int_0^3\frac{1}{3}e^y\,dy=\left.\frac{1}{3}e^y\right]_0^3=\frac{1}{3}(e^3-1)$$

6.

$$\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz=\int_0^1\int_0^z\left[xze^{-y^2}\right]_{x=0}^{x=y}dy\,dz=\int_0^1\int_0^z yze^{-y^2}\,dy\,dz=\int_0^1\left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z}dz$$

$$=\int_0^1-\frac{1}{2}z\left(e^{-z^2}-1\right)dz=\frac{1}{2}\int_0^1\left(z-ze^{-z^2}\right)dz$$

$$=\frac{1}{2}\left[\frac{1}{2}z^2+\frac{1}{2}e^{-z^2}\right]_0^1=\frac{1}{4}(1+e^{-1}-0-1)=\frac{1}{4e}$$

7.

$$\iiint_E 2x\,dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x\,dz\,dx\,dy = \int_0^2 \int_0^{\sqrt{4-y^2}} [2xz]_{z=0}^{z=y} dx\,dy = \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy\,dx\,dy$$

$$= \int_0^2 \left[x^2 y\right]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 (4-y^2) y\,dy = \left[2y^2 - \frac{1}{4} y^4\right]_0^2 = 4$$

8.

$$\iiint_E yz\cos(x^5)\,dV = \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1 \int_0^x \left[\frac{1}{2} yz^2 \cos(x^5)\right]_{z=x}^{z=2x} dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^x 3x^2 y\cos(x^5)\,dy\,dx = \frac{1}{2} \int_0^1 \left[\frac{3}{2} x^2 y^2 \cos(x^5)\right]_{y=0}^{y=x} dx$$

$$= \frac{3}{4} \int_0^1 x^4 \cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20}\sin 1$$

9. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1+x+y\}$ , so

$$\iiint_E 6xy\,dV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2+3x^2y^2+2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (3x^2+3x^3+2x^{5/2})\,dx = \left[x^3+\frac{3}{4}x^4+\frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$$

10. Here $E$ is the region in the first octant that lies below the plane $2x+2y+z=4$ (and above the region in the $xy$ –plane bounded by the lines $x=0$ , $y=0$ , $x+y=2$ ). So

$$\iiint_E y\,dV = \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} y\,dz\,dy\,dx = \int_0^2\int_0^{2-x} y(4-2x-2y)\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} (4y-2xy-2y^2)\,dy\,dx = \int_0^2 \left[2y^2-xy^2-\frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx$$

$$=\int_0^2\left[2(2-x)^2-x(2-x)^2-\frac{2}{3}(2-x)^3\right]dx$$

$$=\int_0^2\left[(2-x)(2-x)^2-\frac{2}{3}(2-x)^3\right]dx=\frac{1}{3}\int_0^2(2-x)^3dx$$

$$=\frac{1}{3}\left[-\frac{1}{4}(2-x)^4\right]_0^2=-\frac{1}{12}(0-16)=\frac{4}{3}$$

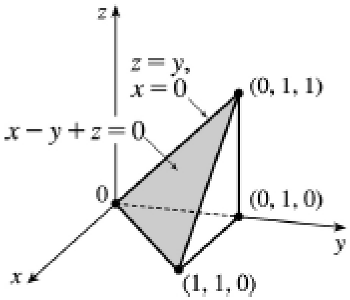
11. Here $E$ is the region that lies below the plane with $x$ –, $y$ –, and $z$ –intercepts 1 , 2 , and 3 respectively, that is, below the plane $2z+6x+3y=6$ and above the region in the $xy$ –plane bounded by the lines $x=0$ , $y=0$ and $6x+3y=6$ . So

$$\iiint_E xy\,dV=\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2}xy\,dz\,dy\,dx=\int_0^1\int_0^{2-2x}\left(3xy-3x^2y-\frac{3}{2}xy^2\right)dy\,dx$$

$$=\int_0^1\left[\frac{3}{2}xy^2-\frac{3}{2}x^2y^2-\frac{1}{2}xy^3\right]_{y=0}^{y=2-2x}dx=\int_0^1(2x-6x^2+6x^3-2x^4)dx$$

$$=\left[x^2-2x^3+\frac{3}{2}x^4-\frac{2}{5}x^5\right]_0^1=\frac{1}{10}\ .$$

12.

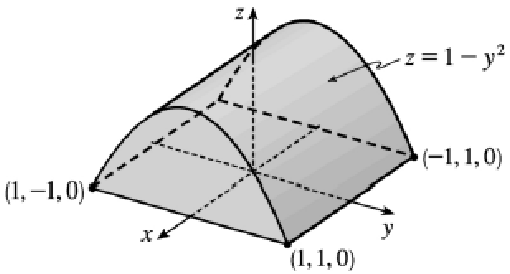



$$\int_0^1\int_0^y\int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1\int_0^y \frac{1}{2}(y-z)^2 z\,dz\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2z^2 - \frac{2}{3}yz^3 + \frac{1}{4}z^4\right]_{z=0}^{z=y} dy$$

$$= \frac{1}{24}\int_0^1 y^4\,dy = \frac{1}{24}\left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{120}$$

13.




$E$ is the region below the parabolic cylinder $z=1-y^2$ and above the square $[-1,1]\times[-1,1]$ in the $xy$ – plane.

$$\iiint_E x^2 e^y \, dV = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2 e^y \, dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} x^2 e^y (1-y^2)\,dy\,dx$$

$$= \int_{-1}^{1} x^2\,dx \int_{-1}^{1} (e^y - y^2 e^y)\,dy$$

$$= \left[\frac{1}{3}x^3\right]_{-1}^{1} \left[e^y - (y^2-2y+2)e^y\right]_{-1}^{1}$$

$$= \frac{1}{3}(2)[e-e-e^{-1}+5e^{-1}] = \frac{8}{3e}$$

**Mass, moment, center of mass and moment of inertia**

3. $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1\}$; $\rho(x, y) = xy^2$

4. $D = \{(x, y) \mid 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$; $\rho(x, y) = cxy$

5. $D$ is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

6. $D$ is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(4, 0)$; $\rho(x, y) = x$

7. $D$ is bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$; $\rho(x, y) = y$

8. $D$ is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$; $\rho(x, y) = x$

9. $D$ is bounded by the parabola $x = y^2$ and the line $y = x - 2$; $\rho(x, y) = 3$

10. $D = \{(x, y) \mid 0 \leqslant y \leqslant \cos x,\ 0 \leqslant x \leqslant \pi/2\}$; $\rho(x, y) = x$

3. $m=\iint_D \rho(x,y)\,dA=\int_0^2\int_{-1}^1 xy^2\,dy\,dx=\int_0^2 x\,dx\int_{-1}^1 y^2\,dy=\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=2\cdot\frac{2}{3}=\frac{4}{3}$ ,

$\bar{x}=\frac{1}{m}\iint_D x\rho(x,y)\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx=\frac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy=\frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=\frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3}=\frac{4}{3}$ ,

$\bar{y}=\frac{1}{m}\iint_D y\rho(x,y)\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx=\frac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy=\frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1=\frac{3}{4}\cdot 2\cdot 0=0$ .

Hence, $(\bar{x},\bar{y})=\left(\frac{4}{3},0\right)$ .

4. $m=\iint_D \rho(x,y)\,dA=\int_0^a\int_0^b cxy\,dy\,dx=c\int_0^a x\,dx\int_0^b y\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{4}a^2b^2c$ ,

$M_y=\iint_D x\rho(x,y)\,dA=\int_0^a\int_0^b cx^2y\,dy\,dx=c\int_0^a x^2\,dx\int_0^b y\,dy=c\left[\frac{1}{3}x^3\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{6}a^3b^2c$ , and

$M_x=\iint_D y\rho(x,y)\,dA=\int_0^a\int_0^b cxy^2\,dy\,dx=c\int_0^a x\,dx\int_0^b y^2\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{3}y^3\right]_0^b=\frac{1}{6}a^2b^3c$ .

Hence, $(\bar{x},\bar{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{2}{3}a,\frac{2}{3}b\right)$ .

5.

$$m=\int_{0}^{2}\int_{x/2}^{3-x}(x+y)\,dy\,dx=\int_{0}^{2}\left[xy+\frac{1}{2}y^{2}\right]_{y=x/2}^{y=3-x}dx=\int_{0}^{2}\left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^{2}-\frac{1}{8}x^{2}\right]dx$$

$$=\int_{0}^{2}\left(-\frac{9}{8}x^{2}+\frac{9}{2}\right)dx=\left[-\frac{9}{8}\left(\frac{1}{3}x^{3}\right)+\frac{9}{2}x\right]_{0}^{2}=6\ ,$$

$$M_{y}=\int_{0}^{2}\int_{x/2}^{3-x}(x^{2}+xy)\,dy\,dx=\int_{0}^{2}\left[x^{2}y+\frac{1}{2}xy^{2}\right]_{y=x/2}^{y=3-x}dx=\int_{0}^{2}\left(\frac{9}{2}x-\frac{9}{8}x^{3}\right)dx=\frac{9}{2}\ \text{, and}$$

$$M_{x}=\int_{0}^{2}\int_{x/2}^{3-y}(xy+y^{2})\,dy\,dx=\int_{0}^{2}\left[\frac{1}{2}xy^{2}+\frac{1}{3}y^{3}\right]_{y=x/2}^{y=3-x}dx=\int_{0}^{2}\left(9-\frac{9}{2}x\right)dx=9\ \text{. Hence } m=6\ ,$$

$$(\bar{x},\bar{y})=\left(\frac{M_{y}}{m},\frac{M_{x}}{m}\right)=\left(\frac{3}{4},\frac{3}{2}\right)\ .$$

6. $m=\int_0^1\int_y^{4-3y} x\,dx\,dy=\int_0^1\left[\frac{1}{2}(4-3y)^2-\frac{1}{2}y^2\right]dy=\left[-\frac{1}{18}(4-3y)^3-\frac{1}{6}y^3\right]_0^1=\frac{10}{3}$ ,

$M_y=\int_0^1\int_y^{4-3y} x^2\,dx\,dy=\int_0^1\left[\frac{1}{3}(4-3y)^3-\frac{1}{3}y^3\right]dy=\left[-\frac{1}{36}(4-3y)^4-\frac{1}{12}y^4\right]_0^1=7$ ,

$M_x=\int_0^1\int_y^{4-3y} xy\,dx\,dy=\int_0^1\left[\frac{1}{2}y(4-3y)^2-\frac{1}{2}y^3\right]dy=\int_0^1(8y-12y^2+4y^3)\,dy=1$ .

Hence $m=\frac{10}{3}$ , $(\bar{x},\bar{y})=(2.1,0.3)$ .

7. $m=\int_0^1\int_0^{e^x} y\,dy\,dx=\int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx=\frac{1}{2}\int_0^1 e^{2x}\,dx=\left.\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{4}(e^2-1)$ ,

$M_y=\int_0^1\int_0^{e^x} xy\,dy\,dx=\frac{1}{2}\int_0^1 xe^{2x}\,dx=\frac{1}{2}\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{8}(e^2+1)$ , and

$M_x=\int_0^1\int_0^{e^x} y^2\,dy\,dx=\int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x}dx=\frac{1}{3}\int_0^1 e^{3x}\,dx=\frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1=\frac{1}{9}(e^3-1)$ .

Hence $m=\frac{1}{4}(e^2-1)$ , $(\bar{x},\bar{y})=\left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right)=\left(\dfrac{e^2+1}{2(e^2-1)},\dfrac{4(e^3-1)}{9(e^2-1)}\right)$ .

8.

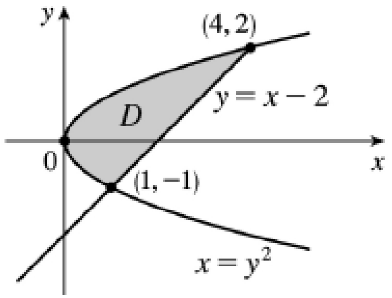
$$m=\int_0^1\int_0^{\sqrt{x}} x\,dy\,dx=\int_0^1 x\,[y]_{y=0}^{y=\sqrt{x}}\,dx=\int_0^1 x^{3/2}\,dx=\left.\frac{2}{5}x^{5/2}\right]_0^1=\frac{2}{5}\ ,$$

$$M_y=\int_0^1\int_0^{\sqrt{x}} x^2\,dy\,dx=\int_0^1 x\,[y]_{y=0}^{y=\sqrt{x}}\,dx=\int_0^1 x^{5/2}\,dx=\left.\frac{2}{7}x^{7/2}\right]_0^1=\frac{2}{7}\ \text{, and}$$

$$M_x=\int_0^1\int_0^{\sqrt{x}} yx\,dy\,dx=\int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}dx=\frac{1}{2}\int_0^1 x^2\,dx=\frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{6}\ .$$

Hence $m=\frac{2}{5}$ , $(\bar{x},\bar{y})=\left(\frac{2/7}{2/5},\frac{1/6}{2/5}\right)=\left(\frac{5}{7},\frac{5}{12}\right)$ .

9.




$$m=\int_{-1}^{2}\int_{y^2}^{y+2} 3\,dx\,dy=\int_{-1}^{2}(3y+6-3y^2)\,dy=\frac{27}{2}\ ,$$

$$M_y = \int_{-1}^{2}\int_{y^2}^{y+2} 3x\,dx\,dy = \int_{-1}^{2} \frac{3}{2}\left[(y+2)^2 - y^4\right]dy$$

$$= \left[\frac{1}{2}(y+2)^3 - \frac{3}{10}y^5\right]_{-1}^{2} = \frac{108}{5}$$

and

$$M_x = \int_{-1}^{2}\int_{y^2}^{y+2} 3y\,dx\,dy = \int_{-1}^{2}(3y^2+6y-3y^3)\,dy$$

$$= \left[y^3+3y^2-\frac{3}{4}y^4\right]_{-1}^{2} = \frac{27}{4}$$

Hence $m = \dfrac{27}{2}$ , $(\bar{x},\bar{y}) = \left(\dfrac{8}{5}, \dfrac{1}{2}\right)$ .

10. $m = \int_{0}^{\pi/2}\int_{0}^{\cos x} x\,dy\,dx = \int_{0}^{\pi/2} x\cos x\,dx = [x\sin x + \cos x]_{0}^{\pi/2} = \dfrac{\pi}{2} - 1$ ,

$M_y = \int_{0}^{\pi/2}\int_{0}^{\cos x} x^2\,dy\,dx = \int_{0}^{\pi/2} x^2\cos x\,dx = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_{0}^{\pi/2} = \dfrac{\pi^2}{4} - 2$ , and

$$M_x = \int_0^{\pi/2}\int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2}\, x\cos^2 x\,dx = \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right]_0^{\pi/2} = \frac{\pi^2}{32} - \frac{1}{8}\ .$$

Hence $m = \dfrac{\pi - 2}{2}$ , $(\bar{x}, \bar{y}) = \left(\dfrac{\pi^2 - 8}{2(\pi - 2)}, \dfrac{\pi + 2}{16}\right)$ .

11. $\rho(x,y) = ky = kr\sin\theta$ , $m = \displaystyle\int_0^{\pi/2}\int_0^1 kr^2 \sin\theta\, dr\, d\theta = \frac{1}{3}k\int_0^{\pi/2} \sin\theta\, d\theta = \frac{1}{3}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{3}k$ ,

$$M_y = \int_0^{\pi/2}\int_0^1 kr^3 \sin\theta\cos\theta\, dr\, d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{8}k\left[-\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}k\ ,$$

$$M_x = \int_0^{\pi/2}\int_0^1 kr^3 \sin^2\theta\, dr\, d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{8}k\left[\theta + \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{16}k\ .$$

Hence $(\bar{x}, \bar{y}) = \left(\dfrac{3}{8}, \dfrac{3\pi}{16}\right)$ .

**15.** Find the moments of inertia $I_x$, $I_y$, $I_0$ for the lamina of Exercise 7.

**16.** Find the moments of inertia $I_x$, $I_y$, $I_0$ for the lamina of Exercise 12.

**17.** Find the moments of inertia $I_x$, $I_y$, $I_0$ for the lamina of Exercise 9.

**18.** Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the $x$-axis or the $y$-axis?

15.

$$I_x = \iint_D y^2 \rho(\mathrm{x,y})\, dA = \int_0^1 \int_0^{e^x} y^2 \cdot y\, dy\, dx = \int_0^1 \left[ \frac{1}{4} y^4 \right]_{y=0}^{y=e^x} dx = \frac{1}{4} \int_0^1 e^{4x} dx$$

$$= \frac{1}{4} \left[ \frac{1}{4} e^{4x} \right]_0^1 = \frac{1}{16} (e^4 - 1) ,$$

$$I_y = \iint_D x^2 \rho(\mathrm{x,y})\, dA = \int_0^1 \int_0^{e^x} x^2 y\, dy\, dx = \int_0^1 x^2 \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 x^2 e^{2x} dx$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}\right)e^{2x}\right]_0^1 \text{ [ integrate by parts twice]}$$

$$=\frac{1}{8}(e^2-1),$$

and $I_0=I_x+I_y=\frac{1}{16}(e^4-1)+\frac{1}{8}(e^2-1)=\frac{1}{16}(e^4+2e^2-3)$ .

16. $I_x=\int_0^{\pi/2}\int_0^1(r^2\sin^2\theta)(kr^2)r\,dr\,d\theta=\frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta=\frac{1}{6}k\left[\frac{1}{4}(2\theta-\sin 2\theta)\right]_0^{\pi/2}=\frac{\pi}{24}k$ ,

$I_y=\int_0^{\pi/2}\int_0^1(r^2\cos^2\theta)(kr^2)r\,dr\,d\theta=\frac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta=\frac{1}{6}k\left[\frac{1}{4}(2\theta+\sin 2\theta)\right]_0^{\pi/2}=\frac{\pi}{24}k$ , and

$I_0=I_x+I_y=\frac{\pi}{12}k$ .

17. $I_x=\int_{-1}^{2}\int_{y^2}^{y+2} 3y^2\,dx\,dy=\int_{-1}^{2}(3y^3+6y^2-3y^4)\,dy=\left[\frac{3}{4}y^4+2y^3-\frac{3}{5}y^5\right]_{-1}^{2}=\frac{189}{20}$ ,

$I_y=\int_{-1}^{2}\int_{y^2}^{y+2} 3x^2\,dx\,dy=\int_{-1}^{2}\left[(y+2)^3-y^6\right]dy=\left[\frac{1}{4}(y+2)^4-\frac{1}{7}y^7\right]_{-1}^{2}=\frac{1269}{28}$ , and $I_0=I_x+I_y=\frac{1917}{35}$ .

18. If we find the moments of inertia about the $x$ – and $y$ –axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the $x$ –axis is given by

$$I_x = \iint_D y^2\rho(x,y)\,dA=\int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx=\int_0^2(1+0.1x)\left[\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$= \frac{8}{3}\int_0^2(1+0.1x)\,dx=\frac{8}{3}\left[x+0.1\cdot\frac{1}{2}x^2\right]_0^2=\frac{8}{3}(2.2)\approx 5.87$$

Similarly, the moment of inertia about the y –axis is given by

$$I_y = \iint_D x^2\rho(x,y)\,dA=\int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx=\int_0^2 x^2(1+0.1x)\,[y]_{y=0}^{y=2}dx$$

$$= 2\int_0^2(x^2+0.1x^3)\,dx=2\left[\frac{1}{3}x^3+0.1\cdot\frac{1}{4}x^4\right]_0^2=2\left(\frac{8}{3}+0.4\right)\approx 6.13$$

Since

$I_y>I_x$ , more force is required to rotate the fan blade about the $y$ –axis.