Revision problems

A-power Series

1-

$$\sum_{n=1}^{\infty} (-1)^n n x^n$$

If $a_n = (-1)^n n x^n$, then

$\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n\, nx^n}\right| = \lim\limits_{n\to\infty} \left|(-1)\frac{n+1}{n}x\right| = \lim\limits_{n\to\infty} \left[\left(1+\frac{1}{n}\right)|x|\right] = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}(-1)^n nx^n$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. Both series $\sum\limits_{n=1}^{\infty}(-1)^n n(\pm 1)^n = \sum\limits_{n=1}^{\infty}(\mp 1)^n n$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty} |(\mp 1)^n n| = \infty$. Thus, the interval of convergence is $I = (-1, 1)$.

2-

$$\sum_{n=1}^{\infty} \frac{X^n}{2n-1}$$

If $a_n = \dfrac{x^n}{2n-1}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{2n+1} \cdot \dfrac{2n-1}{x^n}\right| = \lim\limits_{n\to\infty} \left(\dfrac{2n-1}{2n+1}|x|\right) = \lim\limits_{n\to\infty} \left(\dfrac{2-1/n}{2+1/n}|x|\right) = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{2n-1}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n-1}$ diverges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n}$ since $\dfrac{1}{2n-1} > \dfrac{1}{2n}$ and $\dfrac{1}{2}\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges since it is a constant multiple of the harmonic series. When $x = -1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

3-

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{n+1}\right| = |x| \lim\limits_{n\to\infty} \dfrac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x.

So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

B- Taylor and Maclaurin series

Find the Taylor series for the following problems:

1-

$$f(x) = x^4 - 3x^2 + 1, \quad a = 1$$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0
6	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 5$, so f has a finite series expansion about $a = 1$.

$$\begin{aligned} f(x) = x^4 - 3x^2 + 1 &= \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= \frac{-1}{0!}(x-1)^0 + \frac{-2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 \\ &\quad + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 \\ &= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4 \end{aligned}$$

A finite series converges for all x, so $R = \infty$.

2-

$$f(x) = \cos x, \quad a = \pi$$

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{|x-\pi|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\pi|^{2n}} \right]$$

$$= \lim_{n\to\infty} \frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

Find the Maclaurin series for the following problems:

1-

$$f(x) = (1 - x)^{-2}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1-x)^{-2} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 + 2x + \tfrac{6}{2}x^2 + \tfrac{24}{6}x^3 + \tfrac{120}{24}x^4 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x|\lim_{n\to\infty}\frac{n+2}{n+1} = |x|\,(1) = |x| < 1$$

for convergence, so $R = 1$.

2-

$$f(x) = \sin \pi x$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin \pi x$	0
1	$\pi \cos \pi x$	π
2	$-\pi^2 \sin \pi x$	0
3	$-\pi^3 \cos \pi x$	$-\pi^3$
4	$\pi^4 \sin \pi x$	0
5	$\pi^5 \cos \pi x$	π^5
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\sin \pi x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\
&\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\
&= 0 + \pi x + 0 - \frac{\pi^3}{3!}x^3 + 0 + \frac{\pi^5}{5!}x^5 + \cdots \\
&= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} x^{2n+1}
\end{aligned}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\pi^{2n+3}\, x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\pi^{2n+1}\, x^{2n+1}} \right| = \lim_{n\to\infty} \frac{\pi^2\, x^2}{(2n+3)(2n+2)} = 0 < 1 \quad \text{for all } x \text{, so } R = \infty.$$

Integral

1. $\int_{-2}^{0}(2x+5)\,dx = [x^2+5x]_{-2}^{0} = (0^2+5(0)) - ((-2)^2+5(-2)) = 6$

2. $\int_{-3}^{4}(5-\frac{x}{2})\,dx = \left[5x-\frac{x^2}{4}\right]_{-3}^{4} = \left(5(4)-\frac{4^2}{4}\right) - \left(5(-3)-\frac{(-3)^2}{4}\right) = \frac{133}{4}$

3. $\int_{0}^{4}\left(3x-\frac{x^3}{4}\right)dx = \left[\frac{3x^2}{2}-\frac{x^4}{16}\right]_{0}^{4} = \left(\frac{3(4)^2}{2}-\frac{4^4}{16}\right) - \left(\frac{3(0)^2}{2}-\frac{(0)^4}{16}\right) = 8$

4. $\int_{-2}^{2}(x^3-2x+3)\,dx = \left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2} = \left(\frac{2^4}{4}-2^2+3(2)\right) - \left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right) = 12$

$$\int_{0}^{\pi}\sin x\,dx = [-\cos x]_0^{\pi} = (-\cos\pi) - (-\cos 0) = -(-1) - (-1) = 2$$

$$\int_{0}^{\pi}(1+\cos x)\,dx = [x+\sin x]_0^{\pi} = (\pi+\sin\pi) - (0+\sin 0) = \pi$$

$$\int_{0}^{\pi/3} 2\sec^2 x\,dx = [2\tan x]_0^{\pi/3} = \left(2\tan\left(\frac{\pi}{3}\right)\right) - (2\tan 0) = 2\sqrt{3} - 0 = 2\sqrt{3}$$

$$\int_{\pi/6}^{5\pi/6}\csc^2 x\,dx = [-\cot x]_{\pi/6}^{5\pi/6} = \left(-\cot\left(\frac{5\pi}{6}\right)\right) - \left(-\cot\left(\frac{\pi}{6}\right)\right) = -\left(-\sqrt{3}\right) - \left(-\sqrt{3}\right) = 2\sqrt{3}$$

Indefinite Integrals

1. $\int \sin 3x \, dx, \quad u = 3x$

2. $\int x \sin (2x^2) \, dx, \quad u = 2x^2$

3. $\int \sec 2t \tan 2t \, dt, \quad u = 2t$

4. $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} \, dt, \quad u = 1 - \cos \frac{t}{2}$

1. Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$

   $\int \sin 3x\,dx = \int \frac{1}{3}\sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$

2. Let $u = 2x^2 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$

   $\int x\sin(2x^2)\,dx = \int \frac{1}{4}\sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$

3. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow \frac{1}{2}\,du = dt$

   $\int \sec 2t\tan 2t\,dt = \int \frac{1}{2}\sec u\tan u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2t + C$

4. Let $u = 1 - \cos\frac{t}{2} \Rightarrow du = \frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow 2\,du = \sin\frac{t}{2}\,dt$

   $\int \left(1-\cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)dt = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3 + C$

Double Integrals

Evalute the folowing integrals

1. $\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx$

2. $\int_1^2 \int_y^2 xy\, dx\, dy$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\, dx\, dy$

4. $\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$

6. $\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv$

1.

$$\int_0^1\int_0^{x^2}(x+2y)\,dy\,dx=\int_0^1\left[xy+y^2\right]_{y=0}^{y=x^2}dx=\int_0^1\left[x(x^2)+(x^2)^2-0-0\right]dx$$

$$=\int_0^1(x^3+x^4)\,dx=\left[\frac{1}{4}x^4+\frac{1}{5}x^5\right]_0^1=\frac{9}{20}$$

2.

$$\int_1^2\int_y^2 xy\,dx\,dy=\int_1^2\left[\frac{1}{2}x^2y\right]_{x=y}^{x=2}dy=\int_1^2\frac{1}{2}y(4-y^2)\,dy=\frac{1}{2}\int_1^2(4y-y^3)\,dy$$

$$=\frac{1}{2}\left[2y^2-\frac{1}{4}y^4\right]_1^2=\frac{1}{2}\left(8-4-2+\frac{1}{4}\right)=\frac{9}{8}$$

3.

$$\int_0^1\int_y^{e^y}\sqrt{x}\,dx\,dy=\int_0^1\left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y}dy=\frac{2}{3}\int_0^1(e^{3y/2}-y^{3/2})\,dy=\frac{2}{3}\left[\frac{2}{3}e^{3y/2}-\frac{2}{5}y^{5/2}\right]_0^1$$

$$=\frac{2}{3}\left(\frac{2}{3}e^{3/2}-\frac{2}{5}-\frac{2}{3}e^0+0\right)=\frac{4}{9}e^{3/2}-\frac{32}{45}$$

4.

$$\int_0^1 \int_x^{2-x} (x^2 - y)\,dy\,dx = \int_0^1 \left[x^2 y - \frac{1}{2} y^2 \right]_{y=x}^{y=2-x} dx = \int_0^1 \left[x^2(2-x) - \frac{1}{2}(2-x)^2 - x^2(x) + \frac{1}{2} x^2 \right] dx$$

$$= \int_0^1 (-2x^3 + 2x^2 + 2x - 2)\,dx = \left[-\frac{1}{2} x^4 + \frac{2}{3} x^3 + x^2 - 2x \right]_0^1 = -\frac{5}{6}$$

5.

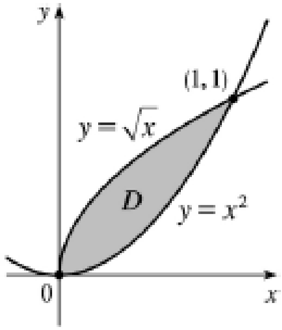
$$\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta = \int_0^{\pi/2} \left[r e^{\sin\theta} \right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\, e^{\sin\theta}\, d\theta = e^{\sin\theta} \Big]_0^{\pi/2}$$

$$= e^{\sin(\pi/2)} - e^0 = e - 1$$

$$\int_0^1\int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \left[u\sqrt{1-v^2}\,\right]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = \left. -\frac{1}{3}(1-v^2)^{3/2}\right]_0^1$$

$$= -\frac{1}{3}(0-1) = \frac{1}{3}$$

14.

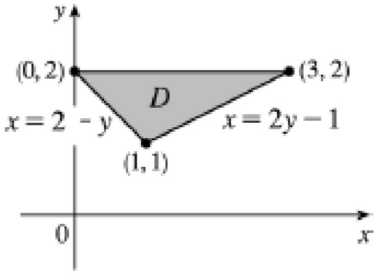


$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2} y^2 \right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{3/2} + \frac{1}{2} x - x^3 - \frac{1}{2} x^4 \right) dx$$

$$= \left[\frac{2}{5} x^{5/2} + \frac{1}{4} x^2 - \frac{1}{4} x^4 - \frac{1}{10} x^5 \right]_0^1 = \frac{3}{10}$$

15.



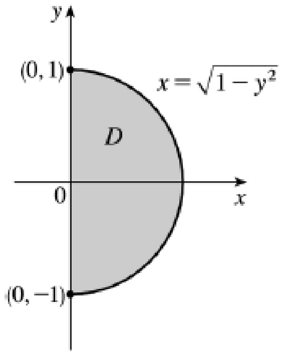
$$\int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 \left[xy^3 \right]_{x=2-y}^{x=2y-1} dy$$

$$= \int_1^2 \left[(2y-1)-(2-y) \right] y^3\,dy$$

$$= \int_1^2 (3y^4 - 3y^3)\,dy = \left[\frac{3}{5}y^5 - \frac{3}{4}y^4 \right]_1^2$$

$$= \frac{96}{5} - 12 - \frac{3}{5} + \frac{3}{4} = \frac{147}{20}$$

16-



$$\iint_D xy^2\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy^2\,dx\,dy$$

$$= \int_{-1}^{1} y^2\left[\frac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \frac{1}{2}\int_{-1}^{1} y^2(1-y^2)\,dy$$

$$= \frac{1}{2}\int_{-1}^{1}(y^2-y^4)\,dy = \frac{1}{2}\left[\frac{1}{3}y^3-\frac{1}{5}y^5\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{5}\right) = \frac{2}{15}$$