Revision problems

A-power Series

1-

$$\sum_{n=1}^{\infty} (-1)^n n x^n$$

If $a_n = (-1)^n n x^n$, then

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n\, n x^n}\right| = \lim_{n\to\infty} \left|(-1)\frac{n+1}{n}x\right| = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)|x|\right] = |x|$. By the Ratio Test, the series $\sum_{n=1}^{\infty} (-1)^n n x^n$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. Both series $\sum_{n=1}^{\infty} (-1)^n n (\pm 1)^n = \sum_{n=1}^{\infty} (\mp 1)^n n$ diverge by the Test for Divergence since $\lim_{n\to\infty} |(\mp 1)^n n| = \infty$. Thus, the interval of convergence is $I = (-1, 1)$.

2-

$$\sum_{n=1}^{\infty} \frac{X^n}{2n-1}$$

If $a_n = \dfrac{x^n}{2n-1}$, then $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2n+1} \cdot \frac{2n-1}{x^n}\right| = \lim_{n\to\infty} \left(\frac{2n-1}{2n+1}|x|\right) = \lim_{n\to\infty} \left(\frac{2-1/n}{2+1/n}|x|\right) = |x|$. By the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ since $\dfrac{1}{2n-1} > \dfrac{1}{2n}$ and $\displaystyle\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is a constant multiple of the harmonic series. When $x = -1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

3-

$$\sum_{n=0}^{\infty} \frac{X^n}{n!}$$

If $a_n = \frac{x^n}{n!}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x.

So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

4-

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 X^n}{2^n}$$

If $a_n = (-1)^n \frac{n^2 x^n}{2^n}$, then

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2\, x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2\, x^n}\right| = \lim_{n\to\infty} \left|\frac{x(n+1)^2}{2n^2}\right| = \lim_{n\to\infty} \left[\frac{|x|}{2}\left(1+\frac{1}{n}\right)^2\right] = \frac{|x|}{2}(1)^2 = \frac{1}{2}|x|$. By the

Ratio Test, the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2x^n}{2^n}$ converges when $\frac{1}{2}|x| < 1 \quad \Leftrightarrow \quad |x| < 2$, so the radius of convergence is $R = 2$.

When $x = \pm 2$, both series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2(\pm 2)^n}{2^n} = \sum_{n=1}^{\infty}(\mp 1)^n n^2$ diverge by the Test for Divergence since

$\lim_{n\to\infty} \left|(\mp 1)^n n^2\right| = \infty$. Thus, the interval of convergence is $I = (-2, 2)$.

5-

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} X^n$$

If $a_n = \frac{(-3)^n x^n}{n^{3/2}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(-3)^n x^n}\right| = \lim_{n\to\infty}\left|-3x\left(\frac{n}{n+1}\right)^{3/2}\right| = 3|x| \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{3/2}$$
$$= 3|x|(1) = 3|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}x^n$ converges when $3|x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{3}$, so $R = \frac{1}{3}$. When $x = \frac{1}{3}$, the series

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$ converges by the Alternating Series Test. When $x = -\frac{1}{3}$, the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series

$\left(p = \frac{3}{2} > 1\right)$. Thus, the interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

6-

$$\sum_{n=2}^{\infty} (-1)^n \frac{X^n}{4^n \ln n}$$

If $a_n = (-1)^n \frac{x^n}{4^n \ln n}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)} \cdot \frac{4^n \ln n}{x^n}\right| = \frac{|x|}{4} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} \cdot 1$

[by l'Hospital's Rule] $= \frac{|x|}{4}$. By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \quad \Leftrightarrow \quad |x| < 4$, so $R = 4$. When

$x = -4$, $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$. Since $\ln n < n$ for $n \geq 2$, $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the

divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$,

$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

7-

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

If $a_n = \dfrac{(x-2)^n}{n^2+1}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \dfrac{n^2+1}{(x-2)^n}\right| = |x-2| \lim\limits_{n\to\infty} \dfrac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty} \dfrac{(x-2)^n}{n^2+1}$ converges when $|x-2| < 1$ $[R=1]$ $\Leftrightarrow$ $-1 < x-2 < 1$ $\Leftrightarrow$ $1 < x < 3$. When $x = 1$, the series $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{1}{n^2+1}$ converges by the Alternating Series Test; when $x = 3$, the series $\sum\limits_{n=0}^{\infty} \dfrac{1}{n^2+1}$ converges by comparison with the p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ $[p = 2 > 1]$. Thus, the interval of convergence is $I = [1, 3]$.

8-

$$\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$$

If $a_n = \dfrac{3^n(x+4)^n}{\sqrt{n}}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{3^n(x+4)^n}\right| = 3\,|x+4|\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 3\,|x+4|$.

By the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{3^n(x+4)^n}{\sqrt{n}}$ converges when $3\,|x+4|<1 \;\Leftrightarrow\; |x+4|<\frac{1}{3} \quad [R=\frac{1}{3}] \;\Leftrightarrow$

$-\frac{1}{3}<x+4<\frac{1}{3} \;\Leftrightarrow\; -\frac{13}{3}<x<-\frac{11}{3}$. When $x=-\frac{13}{3}$, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ converges by the Alternating Series Test; when $x=-\frac{11}{3}$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges $\left[p=\frac{1}{2}\le 1\right]$. Thus, the interval of convergence is $I=\left[-\frac{13}{3},-\frac{11}{3}\right)$.

9-

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^n}$$

If $a_n = \dfrac{(x-2)^n}{n^n}$, then $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test).

$R=\infty$ and $I=(-\infty,\infty)$.

10-

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n, \quad b>0$$

$a_n = \dfrac{n}{b^n}(x-a)^n$, where $b>0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\dfrac{|x-a|}{b}<1 \quad\Leftrightarrow\quad |x-a|<b \quad [\text{so } R=b] \quad\Leftrightarrow\quad -b<x-a<b \quad\Leftrightarrow$

$a-b<x<a+b$. When $|x-a|=b$, $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty} n=\infty$, so the series diverges. Thus, $I=(a-b,a+b)$.

11-

$$\sum_{n=1}^{\infty} n!(2x-1)^n$$

If $a_n = n!\,(2x-1)^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!\,(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim\limits_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$

for all $x\neq\frac{1}{2}$. Since the series diverges for all $x\neq\frac{1}{2}$, $R=0$ and $I=\{\frac{1}{2}\}$.

12-

$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$$

If $a_n = \dfrac{(5x-4)^n}{n^3}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(5x-4)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(5x-4)^n}\right| = \lim_{n\to\infty} |5x-4| \left(\frac{n}{n+1}\right)^3 = \lim_{n\to\infty} |5x-4| \left(\frac{1}{1+1/n}\right)^3$$
$$= |5x-4| \cdot 1 = |5x-4|$$

By the Ratio Test, $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$ converges when $|5x-4| < 1 \iff \left|x - \frac{4}{5}\right| < \frac{1}{5} \iff -\frac{1}{5} < x - \frac{4}{5} < \frac{1}{5} \iff$

$\frac{3}{5} < x < 1$, so $R = \frac{1}{5}$. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$). When $x = \frac{3}{5}$, the series

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = \left[\frac{3}{5}, 1\right]$.

B- Taylor and Maclaurin series

Find the Taylor series for the following problems:

1-

$$f(x) = x^4 - 3x^2 + 1, \quad a = 1$$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0
6	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 5$, so f has a finite series expansion about $a = 1$.

$$f(x) = x^4 - 3x^2 + 1 = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \frac{-1}{0!}(x-1)^0 + \frac{-2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

A finite series converges for all x, so $R = \infty$.

2-

$$f(x) = \ln x, \quad a = 2$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	$1/x$	$1/2$
2	$-1/x^2$	$-1/2^2$
3	$2/x^3$	$2/2^3$
4	$-6/x^4$	$-6/2^4$
5	$24/x^5$	$24/2^5$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{\ln 2}{0!}(x-2)^0 + \frac{1}{1!\,2^1}(x-2)^1 + \frac{-1}{2!\,2^2}(x-2)^2 + \frac{2}{3!\,2^3}(x-2)^3 \\ &\qquad + \frac{-6}{4!\,2^4}(x-2)^4 + \frac{24}{5!\,2^5}(x-2)^5 + \cdots \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{n!\,2^n}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\,2^n}(x-2)^n \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(-1)^{n+1}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-2)n}{(n+1)2}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\frac{|x-2|}{2}$$

$$= \frac{|x-2|}{2} < 1 \quad \text{for convergence, so } |x-2| < 2 \text{ and } R = 2.$$

3-

$$f(x) = e^{2x}, \quad a = 3$$

n	$f^{(n)}(x)$	$f^{(n)}(3)$
0	e^{2x}	e^{6}
1	$2e^{2x}$	$2e^{6}$
2	2^2e^{2x}	$4e^{6}$
3	2^3e^{2x}	$8e^{6}$
4	2^4e^{2x}	$16e^{6}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) = e^{2x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\ &= \frac{e^6}{0!}(x-3)^0 + \frac{2e^6}{1!}(x-3)^1 + \frac{4e^6}{2!}(x-3)^2 \\ &\quad + \frac{8e^6}{3!}(x-3)^3 + \frac{16e^6}{4!}(x-3)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}e^6(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n e^6(x-3)^n}\right| = \lim_{n\to\infty}\frac{2\,|x-3|}{n+1} = 0 < 1 \quad \text{for all } x\text{, so } R = \infty.$$

4-

$f(x) = \cos x, \quad a = \pi$

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{|x-\pi|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\pi|^{2n}} \right]$$

$$= \lim_{n\to\infty} \frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x\text{, so } R = \infty.$$

Find the Maclaurin series for the following problems:

1-

$f(x) = (1 - x)^{-2}$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$(1-x)^{-2} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 + 2x + \tfrac{6}{2}x^2 + \tfrac{24}{6}x^3 + \tfrac{120}{24}x^4 + \cdots$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x| \lim_{n\to\infty}\frac{n+2}{n+1} = |x|\,(1) = |x| < 1$$

for convergence, so $R = 1$.

2-

$f(x) = \sin \pi x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin \pi x$	0
1	$\pi \cos \pi x$	π
2	$-\pi^2 \sin \pi x$	0
3	$-\pi^3 \cos \pi x$	$-\pi^3$
4	$\pi^4 \sin \pi x$	0
5	$\pi^5 \cos \pi x$	π^5
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\sin \pi x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\
&\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\
&= 0 + \pi x + 0 - \frac{\pi^3}{3!}x^3 + 0 + \frac{\pi^5}{5!}x^5 + \cdots \\
&= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} x^{2n+1}
\end{aligned}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\pi^{2n+3}\, x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\pi^{2n+1}\, x^{2n+1}} \right| = \lim_{n\to\infty} \frac{\pi^2 x^2}{(2n+3)(2n+2)} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

3-

$f(x) = 2^x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	2^x	1
1	$2^x(\ln 2)$	$\ln 2$
2	$2^x(\ln 2)^2$	$(\ln 2)^2$
3	$2^x(\ln 2)^3$	$(\ln 2)^3$
4	$2^x(\ln 2)^4$	$(\ln 2)^4$
$\vdots$	$\vdots$	$\vdots$

$$2^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(\ln 2)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(\ln 2)^n x^n} \right|$$

$$= \lim_{n\to\infty} \frac{(\ln 2)\,|x|}{n+1} = 0 < 1 \quad \text{for all } x \text{, so } R = \infty.$$

4-

$f(x) = \sinh x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \quad \text{so } \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Use the Ratio Test to find R. If $a_n = \dfrac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)}$$

$$= 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

Integral

$$\int_{\pi}^{2\pi} \theta \, d\theta = \frac{(2\pi)^2}{2} - \frac{\pi^2}{2} = \frac{3\pi^2}{2}$$

$$\int_{0}^{\sqrt[3]{7}} x^2 \, dx = \frac{\left(\sqrt[3]{7}\right)^3}{3} = \frac{7}{3}$$

$$\int_{0}^{1/2} t^2 \, dt = \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{24}$$

$$\int_{a}^{2a} x \, dx = \frac{(2a)^2}{2} - \frac{a^2}{2} = \frac{3a^2}{2}$$

$$\int_{0}^{\sqrt[3]{b}} x^2 \, dx = \frac{\left(\sqrt[3]{b}\right)^3}{3} = \frac{b}{3}$$

$$\int_{3}^{1} 7 \, dx = 7(1 - 3) = -14$$

$$\int_{0}^{2} 5x \, dx = 5 \int_{0}^{2} x \, dx = 5 \left[\frac{2^2}{2} - \frac{0^2}{2}\right] = 10$$

$$\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \frac{\left(5\sqrt{2}\right)^2}{2} - \frac{\left(\sqrt{2}\right)^2}{2} = 24$$

$$\int_{0}^{0.3} s^2\,ds = \frac{(0.3)^3}{3} = 0.009$$

$$\int_{0}^{\pi/2} \theta^2\,d\theta = \frac{\left(\frac{\pi}{2}\right)^3}{3} = \frac{\pi^3}{24}$$

$$\int_{a}^{\sqrt{3}a} x\,dx = \frac{\left(\sqrt{3}a\right)^2}{2} - \frac{a^2}{2} = a^2$$

$$\int_{0}^{3b} x^2\,dx = \frac{(3b)^3}{3} = 9b^3$$

$$\int_{0}^{-2} \sqrt{2}\,dx = \sqrt{2}(-2-0) = -2\sqrt{2}$$

$$\int_{3}^{5} \frac{x}{8}\,dx = \frac{1}{8}\int_{3}^{5} x\,dx = \frac{1}{8}\left[\frac{5^2}{2} - \frac{3^2}{2}\right] = \frac{16}{16} = 1$$

$$\int_0^2 (2t - 3)\,dt = 2\int_1^1 t\,dt - \int_0^2 3\,dt = 2\left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 3(2 - 0) = 4 - 6 = -2$$

$$\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt = \int_0^{\sqrt{2}} t\,dt - \int_0^{\sqrt{2}} \sqrt{2}\,dt = \left[\frac{\left(\sqrt{2}\right)^2}{2} - \frac{0^2}{2}\right] - \sqrt{2}\left[\sqrt{2} - 0\right] = 1 - 2 = -1$$

$$\int_2^1 \left(1 + \frac{z}{2}\right) dz = \int_2^1 1\,dz + \int_2^1 \frac{z}{2}\,dz = \int_2^1 1\,dz - \frac{1}{2}\int_1^2 z\,dz = 1[1 - 2] - \frac{1}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = -1 - \frac{1}{2}\left(\frac{3}{2}\right) = -\frac{7}{4}$$

$$\int_3^0 (2z - 3)\,dz = \int_3^0 2z\,dz - \int_3^0 3\,dz = -2\int_0^3 z\,dz - \int_3^0 3\,dz = -2\left[\frac{3^2}{2} - \frac{0^2}{2}\right] - 3[0 - 3] = -9 + 9 = 0$$

$$\int_1^2 3u^2\,du = 3\int_1^2 u^2\,du = 3\left[\int_0^2 u^2\,du - \int_0^1 u^2\,du\right] = 3\left(\left[\frac{2^3}{3} - \frac{0^3}{3}\right] - \left[\frac{1^3}{3} - \frac{0^3}{3}\right]\right) = 3\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 3\left(\frac{7}{3}\right) = 7$$

1. $\int_{-2}^{0}(2x+5)\,dx = [x^2+5x]_{-2}^{0} = (0^2+5(0)) - ((-2)^2+5(-2)) = 6$

2. $\int_{-3}^{4}\left(5-\frac{x}{2}\right)dx = \left[5x-\frac{x^2}{4}\right]_{-3}^{4} = \left(5(4)-\frac{4^2}{4}\right) - \left(5(-3)-\frac{(-3)^2}{4}\right) = \frac{133}{4}$

3. $\int_{0}^{4}\left(3x-\frac{x^3}{4}\right)dx = \left[\frac{3x^2}{2}-\frac{x^4}{16}\right]_{0}^{4} = \left(\frac{3(4)^2}{2}-\frac{4^4}{16}\right) - \left(\frac{3(0)^2}{2}-\frac{(0)^4}{16}\right) = 8$

4. $\int_{-2}^{2}(x^3-2x+3)\,dx = \left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2} = \left(\frac{2^4}{4}-2^2+3(2)\right) - \left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right) = 12$

5. $\int_{0}^{1}(x^2+\sqrt{x})\,dx = \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_{0}^{1} = \left(\frac{1}{3}+\frac{2}{3}\right) - 0 = 1$

6. $\int_{0}^{5}x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_{0}^{5} = \frac{2}{5}(5)^{5/2} - 0 = 2(5)^{3/2} = 10\sqrt{5}$

7. $\int_{1}^{32}x^{-6/5}\,dx = \left[-5x^{-1/5}\right]_{1}^{32} = \left(-\frac{5}{2}\right) - (-5) = \frac{5}{2}$

8. $\int_{-2}^{-1}\frac{2}{x^2}\,dx = \int_{-2}^{-1}2x^{-2}\,dx = \left[-2x^{-1}\right]_{-2}^{-1} = \left(\frac{-2}{-1}\right) - \left(\frac{-2}{-2}\right) = 1$

9. $\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2$

10. $\int_0^{\pi} (1 + \cos x) \, dx = [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi$

11. $\int_0^{\pi/3} 2 \sec^2 x \, dx = [2 \tan x]_0^{\pi/3} = \left(2 \tan\left(\frac{\pi}{3}\right)\right) - (2 \tan 0) = 2\sqrt{3} - 0 = 2\sqrt{3}$

12. $\int_{\pi/6}^{5\pi/6} \csc^2 x \, dx = [-\cot x]_{\pi/6}^{5\pi/6} = \left(-\cot\left(\frac{5\pi}{6}\right)\right) - \left(-\cot\left(\frac{\pi}{6}\right)\right) = -\left(-\sqrt{3}\right) - \left(-\sqrt{3}\right) = 2\sqrt{3}$

13. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta \, d\theta = [-\csc \theta]_{\pi/4}^{3\pi/4} = \left(-\csc\left(\frac{3\pi}{4}\right)\right) - \left(-\csc\left(\frac{\pi}{4}\right)\right) = -\sqrt{2} - \left(-\sqrt{2}\right) = 0$

14. $\int_0^{\pi/3} 4 \sec u \tan u \, du = [4 \sec u]_0^{\pi/3} = 4 \sec\left(\frac{\pi}{3}\right) - 4 \sec 0 = 4(2) - 4(1) = 4$

15. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2} \, dt = \int_{\pi/2}^{0} \left(\frac{1}{2} + \frac{1}{2} \cos 2t\right) dt = \left[\frac{1}{2} t + \frac{1}{4} \sin 2t\right]_{\pi/2}^{0} = \left(\frac{1}{2}(0) + \frac{1}{4} \sin 2(0)\right) - \left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4} \sin 2\left(\frac{\pi}{2}\right)\right)$
$= -\frac{\pi}{4}$

16. $\int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2} \, dt = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2} - \frac{1}{2} \cos 2t\right) dt = \left[\frac{1}{2} t - \frac{1}{4} \sin 2t\right]_{-\pi/3}^{\pi/3}$
$= \left(\frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{4} \sin 2\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{2}\left(-\frac{\pi}{3}\right) - \frac{1}{4} \sin 2\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{6} - \frac{1}{4} \sin \frac{2\pi}{3} + \frac{\pi}{6} + \frac{1}{4} \sin\left(\frac{-2\pi}{3}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

17. $\int_{-\pi/2}^{\pi/2}(8y^2+\sin y)\,dy=\left[\frac{8y^3}{3}-\cos y\right]_{-\pi/2}^{\pi/2}=\left(\frac{8\left(\frac{\pi}{2}\right)^3}{3}-\cos\frac{\pi}{2}\right)-\left(\frac{8\left(-\frac{\pi}{2}\right)^3}{3}-\cos\left(-\frac{\pi}{2}\right)\right)=\frac{2\pi^3}{3}$

18. $\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt=\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\pi t^{-2}\right)dt=\left[4\tan t-\frac{\pi}{t}\right]_{-\pi/3}^{-\pi/4}$
$=\left(4\tan\left(-\frac{\pi}{4}\right)-\frac{\pi}{\left(-\frac{\pi}{4}\right)}\right)-\left(4\tan\left(\frac{\pi}{3}\right)-\frac{\pi}{\left(-\frac{\pi}{3}\right)}\right)=(4(-1)+4)-\left(4\left(-\sqrt{3}\right)+3\right)=4\sqrt{3}-3$

19. $\int_{1}^{-1}(r+1)^2\,dr=\int_{1}^{-1}(r^2+2r+1)\,dr=\left[\frac{r^3}{3}+r^2+r\right]_{1}^{-1}=\left(\frac{(-1)^3}{3}+(-1)^2+(-1)\right)-\left(\frac{1^3}{3}+1^2+1\right)=-\frac{8}{3}$

20. $\int_{-\sqrt{3}}^{\sqrt{3}}(t+1)(t^2+4)\,dt=\int_{-\sqrt{3}}^{\sqrt{3}}(t^3+t^2+4t+4)\,dt=\left[\frac{t^4}{4}+\frac{t^3}{3}+2t^2+4t\right]_{-\sqrt{3}}^{\sqrt{3}}$
$=\left(\frac{\left(\sqrt{3}\right)^4}{4}+\frac{\left(\sqrt{3}\right)^3}{3}+2\left(\sqrt{3}\right)^2+4\sqrt{3}\right)-\left(\frac{\left(-\sqrt{3}\right)^4}{4}+\frac{\left(-\sqrt{3}\right)^3}{3}+2\left(-\sqrt{3}\right)^2+4\left(-\sqrt{3}\right)\right)=10\sqrt{3}$

21. $\int_{\sqrt{2}}^{1}\left(\frac{u^7}{2}-\frac{1}{u^5}\right)du=\int_{\sqrt{2}}^{1}\left(\frac{u^7}{2}-u^{-5}\right)du=\left[\frac{u^8}{16}+\frac{1}{4u^4}\right]_{\sqrt{2}}^{1}=\left(\frac{1^8}{16}+\frac{1}{4(1)^4}\right)-\left(\frac{\left(\sqrt{2}\right)^8}{16}+\frac{1}{4\left(\sqrt{2}\right)^4}\right)=-\frac{3}{4}$

22. $\int_{1/2}^{1}\left(\frac{1}{v^3}-\frac{1}{v^4}\right)dv=\int_{1/2}^{1}(v^{-3}-v^{-4})\,dv=\left[\frac{-1}{2v^2}+\frac{1}{3v^3}\right]_{1/2}^{1}=\left(\frac{-1}{2(1)^2}+\frac{1}{3(1)^3}\right)-\left(\frac{-1}{2\left(\frac{1}{2}\right)^2}+\frac{1}{3\left(\frac{1}{2}\right)^3}\right)=-\frac{5}{6}$

23. $\int_{1}^{\sqrt{2}}\frac{s^2+\sqrt{s}}{s^2}\,ds=\int_{1}^{\sqrt{2}}\left(1+s^{-3/2}\right)ds=\left[s-\frac{2}{\sqrt{s}}\right]_{1}^{\sqrt{2}}=\left(\sqrt{2}-\frac{2}{\sqrt{\sqrt{2}}}\right)-\left(1-\frac{2}{\sqrt{1}}\right)=\sqrt{2}-2^{3/4}+1$
$=\sqrt{2}-\sqrt[4]{8}+1$

Indefinite Integrals

1. $\int \sin 3x \, dx, \quad u = 3x$

2. $\int x \sin(2x^2) \, dx, \quad u = 2x^2$

3. $\int \sec 2t \tan 2t \, dt, \quad u = 2t$

4. $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} \, dt, \quad u = 1 - \cos \frac{t}{2}$

5. $\int 28(7x - 2)^{-5} \, dx, \quad u = 7x - 2$

6. $\int x^3(x^4 - 1)^2 \, dx, \quad u = x^4 - 1$

7. $\int \frac{9r^2 \, dr}{\sqrt{1 - r^3}}, \quad u = 1 - r^3$

8. $\int 12(y^4 + 4y^2 + 1)^2(y^3 + 2y) \, dy, \quad u = y^4 + 4y^2 + 1$

9. $\int \sqrt{x} \sin^2(x^{3/2} - 1) \, dx, \quad u = x^{3/2} - 1$

10. $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right) dx, \quad u = -\frac{1}{x}$

11. $\displaystyle\int \csc^2 2\theta \cot 2\theta \, d\theta$

a. Using $u = \cot 2\theta$ **b.** Using $u = \csc 2\theta$

12. $\displaystyle\int \frac{dx}{\sqrt{5x + 8}}$

a. Using $u = 5x + 8$ **b.** Using $u = \sqrt{5x + 8}$

1. Let $u = 3x \;\Rightarrow\; du = 3\,dx \;\Rightarrow\; \frac{1}{3}\,du = dx$

$$\int \sin 3x\,dx = \int \frac{1}{3}\sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$$

2. Let $u = 2x^2 \;\Rightarrow\; du = 4x\,dx \;\Rightarrow\; \frac{1}{4}\,du = x\,dx$

$$\int x\sin(2x^2)\,dx = \int \frac{1}{4}\sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$$

3. Let $u = 2t \;\Rightarrow\; du = 2\,dt \;\Rightarrow\; \frac{1}{2}\,du = dt$

$$\int \sec 2t\tan 2t\,dt = \int \frac{1}{2}\sec u\tan u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2t + C$$

4. Let $u = 1 - \cos\frac{t}{2} \;\Rightarrow\; du = \frac{1}{2}\sin\frac{t}{2}\,dt \;\Rightarrow\; 2\,du = \sin\frac{t}{2}\,dt$

$$\int \left(1 - \cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)dt = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$$

5. Let $u = 7x - 2 \;\Rightarrow\; du = 7\,dx \;\Rightarrow\; \frac{1}{7}\,du = dx$

$$\int 28(7x-2)^{-5}\,dx = \int \frac{1}{7}(28)u^{-5}\,du = \int 4u^{-5}\,du = -u^{-4} + C = -(7x-2)^{-4} + C$$

6. Let $u = x^4 - 1 \;\Rightarrow\; du = 4x^3\,dx \;\Rightarrow\; \frac{1}{4}\,du = x^3\,dx$

$$\int x^3\left(x^4 - 1\right)^2 dx = \int \frac{1}{4}u^2\,du = \frac{u^3}{12} + C = \frac{1}{12}\left(x^4 - 1\right)^3 + C$$

7. Let $u = 1 - r^3 \Rightarrow du = -3r^2\, dr \Rightarrow -3\, du = 9r^2\, dr$

$$\int \frac{9r^2\, dr}{\sqrt{1-r^3}} = \int -3u^{-1/2}\, du = -3(2)u^{1/2} + C = -6\left(1 - r^3\right)^{1/2} + C$$

8. Let $u = y^4 + 4y^2 + 1 \Rightarrow du = (4y^3 + 8y)\, dy \Rightarrow 3\, du = 12\,(y^3 + 2y)\, dy$

$$\int 12\,(y^4 + 4y^2 + 1)^2\,(y^3 + 2y)\, dy = \int 3u^2\, du = u^3 + C = \left(y^4 + 4y^2 + 1\right)^3 + C$$

9. Let $u = x^{3/2} - 1 \Rightarrow du = \frac{3}{2}x^{1/2}\, dx \Rightarrow \frac{2}{3}\, du = \sqrt{x}\, dx$

$$\int \sqrt{x}\, \sin^2\left(x^{3/2} - 1\right) dx = \int \frac{2}{3}\sin^2 u\, du = \frac{2}{3}\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = \frac{1}{3}\left(x^{3/2} - 1\right) - \frac{1}{6}\sin\left(2x^{3/2} - 2\right) + C$$

10. Let $u = -\frac{1}{x} \Rightarrow du = \frac{1}{x^2}\, dx$

$$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right) dx = \int \cos^2(-u)\, du = \int \cos^2(u)\, du = \left(\frac{u}{2} + \frac{1}{4}\sin 2u\right) + C = -\frac{1}{2x} + \frac{1}{4}\sin\left(-\frac{2}{x}\right) + C$$
$$= -\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$$

11. (a) Let $u = \cot 2\theta \Rightarrow du = -2\csc^2 2\theta\, d\theta \Rightarrow -\frac{1}{2}\, du = \csc^2 2\theta\, d\theta$

$$\int \csc^2 2\theta \cot 2\theta\, d\theta = -\int \frac{1}{2}u\, du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$$

(b) Let $u = \csc 2\theta \Rightarrow du = -2\csc 2\theta \cot 2\theta\, d\theta \Rightarrow -\frac{1}{2}\, du = \csc 2\theta \cot 2\theta\, d\theta$

$$\int \csc^2 2\theta \cot 2\theta\, d\theta = \int -\frac{1}{2}u\, du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$$

12. (a) Let $u = 5x + 8 \Rightarrow du = 5\, dx \Rightarrow \frac{1}{5}\, du = dx$

$$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{5}\left(\frac{1}{\sqrt{u}}\right) du = \frac{1}{5}\int u^{-1/2}\, du = \frac{1}{5}\left(2u^{1/2}\right) + C = \frac{2}{5}\, u^{1/2} + C = \frac{2}{5}\sqrt{5x+8} + C$$

(b) Let $u = \sqrt{5x+8} \Rightarrow du = \frac{1}{2}(5x+8)^{-1/2}(5)\, dx \Rightarrow \frac{2}{5}\, du = \frac{dx}{\sqrt{5x+8}}$

$$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{2}{5}\, du = \frac{2}{5}\, u + C = \frac{2}{5}\sqrt{5x+8} + C$$

Double Integrals

Evalute the folowing integrals

1. $\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx$

2. $\int_1^2 \int_y^2 xy\, dx\, dy$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\, dx\, dy$

4. $\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$

6. $\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv$

7. $\iint_D x^3y^2\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ -x \leqslant y \leqslant x\}$

8. $\iint_D \frac{4y}{x^3 + 2}\, dA, \quad D = \{(x, y) \mid 1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2x\}$

9. $\iint_D \frac{2y}{x^2 + 1}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant \sqrt{x}\}$

10. $\iint_D e^{y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$

11. $\iint_D e^{x/y}\, dA, \quad D = \{(x, y) \mid 1 \leqslant y \leqslant 2,\ y \leqslant x \leqslant y^3\}$

12. $\iint_D x\sqrt{y^2 - x^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$

13. $\iint_D x \cos y \, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

14. $\iint_D (x + y) \, dA$, D is bounded by $y = \sqrt{x}$ and $y = x^2$

15. $\iint_D y^3 \, dA$,

D is the triangular region with vertices (0, 2), (1, 1), and (3, 2)

16. $\iint_D xy^2 \, dA$, D is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$

17. $\iint_D (2x - y) \, dA$,

D is bounded by the circle with center the origin and radius 2

18. $\iint_D 2xy \, dA$, D is the triangular region with vertices (0, 0),

(1, 2), and (0, 3)

1.

$$\int_0^1\int_0^{x^2}(x+2y)\,dy\,dx=\int_0^1\left[xy+y^2\right]_{y=0}^{y=x^2}dx=\int_0^1\left[x(x^2)+(x^2)^2-0-0\right]dx$$

$$=\int_0^1(x^3+x^4)\,dx=\left[\frac{1}{4}x^4+\frac{1}{5}x^5\right]_0^1=\frac{9}{20}$$

2.

$$\int_1^2\int_y^2 xy\,dx\,dy=\int_1^2\left[\frac{1}{2}x^2y\right]_{x=y}^{x=2}dy=\int_1^2\frac{1}{2}y(4-y^2)\,dy=\frac{1}{2}\int_1^2(4y-y^3)\,dy$$

$$=\frac{1}{2}\left[2y^2-\frac{1}{4}y^4\right]_1^2=\frac{1}{2}\left(8-4-2+\frac{1}{4}\right)=\frac{9}{8}$$

3.

$$\int_0^1\int_y^{e^y}\sqrt{x}\,dx\,dy=\int_0^1\left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y}dy=\frac{2}{3}\int_0^1(e^{3y/2}-y^{3/2})\,dy=\frac{2}{3}\left[\frac{2}{3}e^{3y/2}-\frac{2}{5}y^{5/2}\right]_0^1$$

$$=\frac{2}{3}\left(\frac{2}{3}e^{3/2}-\frac{2}{5}-\frac{2}{3}e^0+0\right)=\frac{4}{9}e^{3/2}-\frac{32}{45}$$

4.

$$\int_0^1\int_x^{2-x}(x^2-y)\,dy\,dx=\int_0^1\left[x^2y-\frac{1}{2}y^2\right]_{y=x}^{y=2-x}dx=\int_0^1\left[x^2(2-x)-\frac{1}{2}(2-x)^2-x^2(x)+\frac{1}{2}x^2\right]dx$$

$$=\int_0^1(-2x^3+2x^2+2x-2)\,dx=\left[-\frac{1}{2}x^4+\frac{2}{3}x^3+x^2-2x\right]_0^1=-\frac{5}{6}$$

5.

$$\int_0^{\pi/2}\int_0^{\cos\theta}e^{\sin\theta}\,dr\,d\theta=\int_0^{\pi/2}\left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta}d\theta=\int_0^{\pi/2}(\cos\theta)\,e^{\sin\theta}\,d\theta=e^{\sin\theta}\Big]_0^{\pi/2}$$

$$=e^{\sin(\pi/2)}-e^0=e-1$$

$$\int_0^1\int_0^v \sqrt{1-v^2}\,du\,dv=\int_0^1\left[u\sqrt{1-v^2}\right]_{u=0}^{u=v}dv=\int_0^1 v\sqrt{1-v^2}\,dv=\left.-\frac{1}{3}(1-v^2)^{3/2}\right]_0^1$$

$$=-\frac{1}{3}(0-1)=\frac{1}{3}$$

7.

$$\iint_D x^3y^2\,dA=\int_0^2\int_{-x}^{x} x^3y^2\,dy\,dx=\int_0^2\left[\frac{1}{3}x^3y^3\right]_{y=-x}^{y=x}dx=\frac{1}{3}\int_0^2 2x^6\,dx$$

$$=\frac{2}{3}\left[\frac{1}{7}x^7\right]_0^2=\frac{2}{21}\left[2^7-0\right]=\frac{256}{21}$$

8.

$$\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2\int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2 \left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$$

$$= \frac{8}{3}\ln\left|x^3+2\right|\Big]_1^2 = \frac{8}{3}(\ln 10 - \ln 3) = \frac{8}{3}\ln\frac{10}{3}$$

9.

$$\int_0^1\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+1\right|\Big]_0^1 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

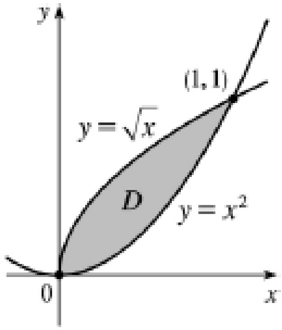
10. $$\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left[x e^{y^2}\right]_{x=0}^{x=y} dy = \int_0^1 y e^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big]_0^1 = \frac{1}{2}(e-1)$$

$$11.\ \int_1^2\int_y^{y^3} e^{x/y}\,dx\,dy=\int_1^2\left[ye^{x/y}\right]_{x=y}^{x=y^3}dy=\int_1^2\left(ye^{y^2}-ey\right)dy=\left[\frac{1}{2}e^{y^2}-\frac{1}{2}ey^2\right]_1^2=\frac{1}{2}(e^4-4e)$$

$$12.\ \int_0^1\int_0^y x\sqrt{y^2-x^2}\,dx\,dy=\int_0^1\left[-\frac{1}{3}(y^2-x^2)^{3/2}\right]_{x=0}^{x=y}dy=\frac{1}{3}\int_0^1 y^3\,dy=\quad \frac{1}{3}\cdot\frac{1}{4}y^4\Big]_0^1=\frac{1}{12}$$

$$13.\ \int_0^1\int_0^{x^2} x\cos y\,dy\,dx=\int_0^1\left[x\sin y\right]_{y=0}^{y=x^2}dx=\int_0^1 x\sin x^2\,dx=\quad -\frac{1}{2}\cos x^2\Big]_0^1=\frac{1}{2}(1-\cos 1)$$

14.

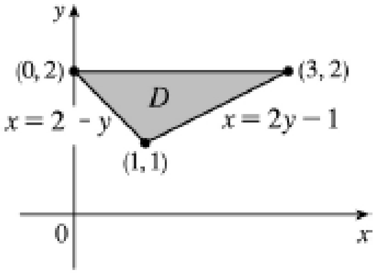


$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2} y^2 \right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{3/2} + \frac{1}{2} x - x^3 - \frac{1}{2} x^4 \right) dx$$

$$= \left[\frac{2}{5} x^{5/2} + \frac{1}{4} x^2 - \frac{1}{4} x^4 - \frac{1}{10} x^5 \right]_0^1 = \frac{3}{10}$$

15.



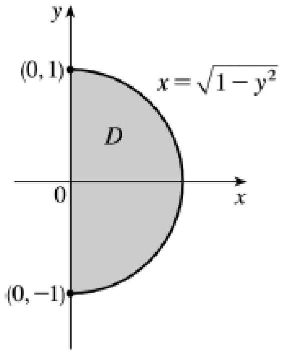
$$\int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 \left[xy^3\right]_{x=2-y}^{x=2y-1} dy$$

$$= \int_1^2 \left[(2y-1)-(2-y)\right] y^3\,dy$$

$$= \int_1^2 (3y^4-3y^3)\,dy = \left[\frac{3}{5}y^5 - \frac{3}{4}y^4\right]_1^2$$

$$= \frac{96}{5} - 12 - \frac{3}{5} + \frac{3}{4} = \frac{147}{20}$$

16-



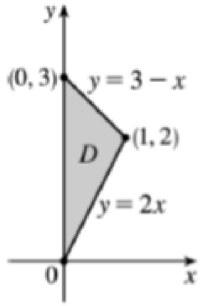
$$\iint_D xy^2\,dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy^2\,dx\,dy$$

$$= \int_{-1}^{1} y^2\left[\frac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \frac{1}{2}\int_{-1}^{1} y^2(1-y^2)\,dy$$

$$= \frac{1}{2}\int_{-1}^{1}(y^2-y^4)\,dy = \frac{1}{2}\left[\frac{1}{3}y^3-\frac{1}{5}y^5\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{5}\right) = \frac{2}{15}$$

17-



$$\iint_D xy\,dA = \int_0^1 \int_{2x}^{3-x} 2xy\,dy\,dx = \int_0^1 \left[xy^2\right]_{y=2x}^{y=3-x} dx$$

$$= \int_0^1 x[(3-x)^2 - (2x)^2]\,dx$$

$$= \int_0^1 (-3x^3 - 6x^2 + 9x)\,dx$$

$$= \left[-\frac{3}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^1 = -\frac{3}{4} - 2 + \frac{9}{2} = \frac{7}{4}$$