Tutorial No. 10 Solution

Integral

1-

Find $\int \cos^2 ax\, dx$.

$$\int \cos^2 ax\, dx = \int \frac{1 + \cos 2ax}{2}\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{a} \sin ax \cos ax\right) + C.$$

$$= \frac{1}{2}\left(x + \frac{1}{2a} \sin 2ax\right) + C$$

2-

Find $\int \sin^2 ax\, dx$.

$$\int \sin^2 ax\, dx = \int (1 - \cos^2 ax)\, dx$$

$$= x - \frac{1}{2}\left(x + \frac{1}{a} \sin ax \cos ax\right) + C$$

$$\frac{1}{2}\left(x - \frac{1}{a} \sin ax \cos ax\right) + C.$$

3- Find

$\int \sin x \cos^2 x \, dx.$

Let $u = \cos x,$

$du = -\sin x \, dx.$

Then $\int \sin x \cos^2 dx$

$= -\int u^2 \, du = -\frac{1}{3}u^3 + C$

$= -\frac{1}{3} \cos^3 x + C.$

4- Find

$\int \sin^4 x \cos^5 x \, dx.$

Since the power of $\cos x$ is odd, let $u = \sin x,$

$du = \cos x \, dx.$

Then $\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x(1 - \sin^2 x)^2 \cos x \, dx$

$= \int u^4(1 - u^2)^2 \, du = \int u^4(1 - 2u^2 + u^4) \, du$

$u^5(\frac{1}{5} - \frac{2}{7}u^2 + \frac{1}{9}u^4) + C = \sin^5 x(\frac{1}{5} - \frac{2}{7} \sin^2 x + \frac{1}{9} \sin^4 x) + C.$

$= \int (u^4 - 2u^6 + u^8) \, du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$

$= \sin^5 x(\frac{1}{5} - \frac{2}{7} \sin^2 x + \frac{1}{9} \sin^4 x) + C.$

5- Find

$\int \cos^6 x\, dx.$

$\int \cos^6 x\, dx = \int (\cos^2 x)^3\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx = \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\, dx = \frac{1}{8}[x + \frac{3}{2}\sin 2x + 3\int \frac{1+\cos 4x}{2}\, dx + \int (1-\sin^2 2x)\cos 2x\, dx]$. Now, $\int \frac{1+\cos 4x}{2}\, dx = \frac{1}{2}(x + \frac{1}{4}\sin 4x) = \frac{1}{2}(x + \frac{1}{2}\sin 2x \cos 2x)$. Also, in $\int (1-\sin^2 2x)\cos 2x\, dx$, let $u = \sin 2x$, $du = 2\cos 2x\, dx$. So we get $\frac{1}{2}\int (1-u^2)\, du = \frac{1}{2}(u - \frac{1}{3}u^3) = \frac{1}{6}u(3-2u^3) = \frac{1}{6}\sin 2x(3 - 2\sin^3 2x)$. Hence, the entire answer is $\frac{1}{8}[x + \frac{3}{2}\sin 2x + \frac{3}{2}(x + \frac{1}{2}\sin 2x\cos 2x) + \frac{1}{6}\sin 2x(3 - 2\sin^3 2x)] + C = \frac{1}{8}[\frac{5}{2}x + 2\sin 2x + \frac{3}{4}\sin 2x\cos 2x - \frac{1}{3}\sin^4 2x] + C$.

6- Find

$\int \cos^4 x \sin^2 x\, dx.$

$$\int \cos^4 x \sin^2 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int (1 + 2\cos 2x + \cos^2 2x)(1-\cos 2x)\, dx = \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}(x + \frac{1}{2}\sin 2x - \int \cos^2 2x\, dx - \int \cos^3 2x\, dx)$$

Now, $\int \cos^2 2x\, dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x \cos 2x)$

$$\int \cos^3 2x\, dx = \int (1-\sin^2 2x)\cos 2x\, dx$$

$$= \int \cos 2x\, dx - \int \sin^2 2x \cos 2x\, dx$$

$$= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x$$

Hence, we get $\frac{1}{8}[x + \frac{1}{2}\sin 2x - \frac{1}{2}(x + \frac{1}{2}\sin 2x\cos 2x) + \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x] + C$

$$= \frac{1}{8}[(x/2) + \sin 2x - \frac{1}{4}\sin 2x\cos 2x - \frac{1}{6}\sin^3 2x] + C.$$

7-

$$\int \tan^2 \frac{x}{2}\, dx.$$

Let $x = 2u, \quad dx = 2\, du.$

Then $\int \tan^2 \frac{x}{2}\, dx = 2 \int \tan^2 u\, du$

$$= 2 \int (\sec^2 u - 1)\, du = 2(\tan u - u) + C$$

$$2\left(\tan \frac{x}{2} - \frac{x}{2}\right) + C = 2 \tan \frac{x}{2} - x + C.$$

8-

$$\int \tan^4 x\, dx.$$

$$\int \tan^4 x\, dx = \int \tan^2 x\, (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \tfrac{1}{3} \tan^3 x - \tan x + x + C$$

$$= \tfrac{1}{3} \tan^3 x - \int (\sec^2 x - 1)\, dx$$

$$= \tfrac{1}{3} \tan^3 x - \tan x + x + C$$

9-

$$\int \sec^5 x\, dx.$$

$$\int \sec^5 x\, dx = \frac{\tan x \sec^3 x}{4} + \frac{3}{4} \int \sec^3 x\, dx = \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\left(\frac{\tan x \sec x}{2} + \frac{1}{3} \int \sec x\, dx\right)$$

$$= \frac{\tan x \sec^3 x}{4} + \frac{3}{8} \tan x \sec x + \frac{1}{4} \ln |\sec x + \tan x| + C$$

10-

$\int \tan^2 x \sec^4 x\, dx.$

Since the exponent of $\sec x$ is even, $\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x\ (1 + \tan^2 x) \sec^2 x\, dx = \int (\tan^2 x \sec^2 x + \tan^4 x \sec^2 x)\, dx = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C.$

11-

$\int \tan^3 x \sec^3 x\, dx.$

Since the exponent of $\tan x$ is odd,

$\int \tan^3 x \sec^3 dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x\, dx$

$= \int (\sec^4 x \sec x \tan x - \sec^2 x \sec x \tan x)\, dx$

$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C.$

12-

$\int \tan^4 x \sec x\, dx.$

$\int \tan^4 x \sec x\, dx \quad = \quad \int (\sec^2 x - 1)^2 \sec x\, dx$

$= \quad \int (\sec^4 x - 2 \sec^2 x + 1) \sec x\, dx$

$= \quad \int (\sec^5 x - 2 \sec^3 x + \sec x)\, dx.$

$\int \sec^5 x\, dx = \dfrac{\tan x \sec^3 x}{4} + \dfrac{3}{4} \int \sec^3 x\, dx,$ and $\int \sec^3 x\, dx =$

$\dfrac{1}{2} (\tan x \sec x + \ln |\sec x + \tan x|).$

Thus, we get $\dfrac{\tan x \sec^3 x}{4} + \dfrac{3}{4}\displaystyle\int \sec^3 x\, dx - 2\int \sec^3 x\, dx + \ln|\sec x +$

$$\tan x| = \frac{\tan x \sec^3 x}{4} - \frac{5}{4}\int \sec^3 x\, dx + \ln|\sec x + \tan x|$$

$$= \frac{\tan x \sec^3 x}{4} - \frac{5}{4}\left(\frac{1}{2}\tan x \sec x + \ln|\sec x + \tan x|\right) +$$

$$\ln|\sec x + \tan x| + C = \frac{\tan x \sec^3 x}{4} - \frac{5}{8}\tan x \sec x + \frac{3}{8}\ln|\sec x + \tan x| + C.$$