Dr. Mohamed El-Shazly
CALCULUS FOR ENGINEERS
MATH 1110

Tutorial No. 12Solution

Application of Double Integral

Evaluate the following iterated integral

1-

Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$$

$$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$$

2-

Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

$$V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2\, dy\, dx = \int_{-2}^{1}[x^2 y]_{x}^{2-x^2}\, dx$$

$$= \int_{-2}^{1}(2x^2 - x^4 - x^3)\, dx = \left[\tfrac{2}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{4}x^4\right]_{-2}^{1}$$

$$= \left(\tfrac{2}{3} - \tfrac{1}{5} - \tfrac{1}{4}\right) - \left(-\tfrac{16}{3} + \tfrac{32}{5} - \tfrac{16}{4}\right)$$

$$= \left(\tfrac{40}{60} - \tfrac{12}{60} - \tfrac{15}{60}\right) - \left(-\tfrac{320}{60} + \tfrac{384}{60} - \tfrac{240}{60}\right) = \tfrac{189}{60} = \tfrac{63}{20}$$

3-

Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

$$V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x + 4)\, dy\, dx$$

$$= \int_{-4}^{1}[xy + 4y]_{3x}^{4-x^2}\, dx =$$

$$= \int_{-4}^{1}[x(4 - x^2) + 4(4 - x^2) - 3x^2 - 12x]\, dx$$

$$= \int_{-4}^{1}(-x^3 - 7x^2 - 8x + 16)\,dx$$

$$= \left[-\tfrac{1}{4}x^4 - \tfrac{7}{3}x^3 - 4x^2 + 16x\right]_{-4}^{1}$$

$$= \left(-\tfrac{1}{4} - \tfrac{7}{3} + 12\right) - \left(\tfrac{64}{3} - 64\right)$$

$$= \tfrac{157}{3} - \tfrac{1}{4} = \tfrac{625}{12}$$

4-

Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3 - y)\,dy\,dx$$

$$= \int_0^2 \left[3y - \tfrac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left[3\sqrt{4 - x^2} - \left(\tfrac{4-x^2}{2}\right)\right] dx$$

$$= \left[\tfrac{3}{2}x\sqrt{4 - x^2} + 6\sin^{-1}\left(\tfrac{x}{2}\right) - 2x + \tfrac{x^3}{6}\right]_0^2$$

$$= 6\left(\tfrac{\pi}{2}\right) - 4 + \tfrac{8}{6} = 3\pi - \tfrac{16}{6} = \tfrac{9\pi-8}{3}$$

5-

Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

$$V = \int_0^2 \int_0^3 (4 - y^2)\, dx\, dy$$

$$= \int_0^2 [4x - y^2 x]_0^3\, dy$$

$$= \int_0^2 (12 - 3y^2)\, dy$$

$$= [12y - y^3]_0^2 = 24 - 8 = 16$$

6-

Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

$$V = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\, dy\, dx$$

$$= \int_0^2 \left[(4 - x^2)\, y - \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_0^2 \frac{1}{2} (4 - x^2)^2\, dx = \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx$$

$$= \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = \frac{480-320+96}{30} = \frac{128}{15}$$

7-

Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

$$V = \int_0^2 \int_0^{2-x} (12 - 3y^2)\, dy\, dx$$

$$= \int_0^2 [12y - y^3]_0^{2-x}\, dx$$

$$= \int_0^2 [24 - 12x - (2 - x)^3]\, dx$$

$$= \left[24x - 6x^2 + \frac{(2-x)^4}{4}\right]_0^2 = 20$$