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CALCULUS FOR ENGINEERS
MATH 1110

Tutorial No. 13 Solution

Triple integrals

Evaluate the following iterated integral

1-

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz\; dy\; dx\; dz$$

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1 \int_0^z [6xyz]_{y=0}^{y=x+z}\,dx\,dz = \int_0^1 \int_0^z 6xz(x+z)\,dx\,dz$$

$$= \int_0^1 \left[2x^3z + 3x^2z^2\right]_{x=0}^{x=z} dz = \int_0^1 (2z^4 + 3z^4)\,dz = \int_0^1 5z^4\,dz = z^5\Big]_0^1 = 1$$

2-

$$\int_0^1 \int_x^{2x} \int_0^{y} 2xyz\; dz\; dy\; dx$$

$$\int_0^1 \int_x^{2x} \int_0^{y} 2xyz\,dz\,dy\,dx = \int_0^1 \int_x^{2x} \left[xyz^2\right]_{z=0}^{z=y} dy\,dx = \int_0^1 \int_x^{2x} xy^3\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x} dx = \int_0^1 \frac{15}{4}x^5\,dx = \left.\frac{5}{8}x^6\right]_0^1 = \frac{5}{8}$$

3-

$$\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y \, dx \, dz \, dy$$

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy = \int_0^3\int_0^1 \left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}} dz\,dy = \int_0^3\int_0^1 ze^y\sqrt{1-z^2}\,dz\,dy$$

$$= \int_0^3 \left[-\frac{1}{3}(1-z^2)^{3/2} e^y\right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3} e^y \, dy = \left.\frac{1}{3} e^y\right]_0^3 = \frac{1}{3}(e^3-1)$$

4-

$$\int_0^1 \int_0^z \int_0^y ze^{-y^2} \, dx \, dy \, dz$$

$$\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1\int_0^z \left[xze^{-y^2}\right]_{x=0}^{x=y} dy\,dz = \int_0^1\int_0^z yze^{-y^2}\,dy\,dz = \int_0^1 \left[-\frac{1}{2} ze^{-y^2}\right]_{y=0}^{y=z} dz$$

$$= \int_0^1 -\frac{1}{2} z\left(e^{-z^2}-1\right) dz = \frac{1}{2}\int_0^1 \left(z - ze^{-z^2}\right) dz$$

$$= \frac{1}{2}\left[\frac{1}{2}z^2 + \frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}(1+e^{-1}-0-1) = \frac{1}{4e}$$

5-

$\iiint_E 6xy \, dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1+x+y\}$, so

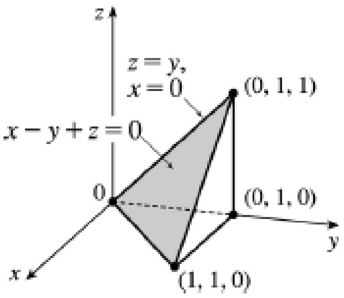
$$\iiint_E 6xy\,dV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}} [6xyz]_{z=0}^{z=1+x+y}\,dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2+3x^2y^2+2xy^3\right]_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \int_0^1 (3x^2+3x^3+2x^{5/2})\,dx = \left[x^3+\frac{3}{4}x^4+\frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$$

6-

$\iiint_E xz\,dV$, where E is the solid tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$



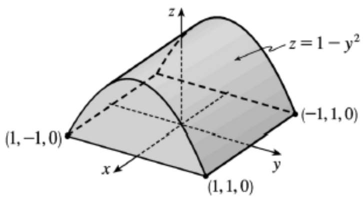
$$\int_0^1\int_0^y\int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1\int_0^y \frac{1}{2}(y-z)^2 z\,dz\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2z^2 - \frac{2}{3}yz^3 + \frac{1}{4}z^4\right]_{z=0}^{z=y} dy$$

$$= \frac{1}{24}\int_0^1 y^4\,dy = \frac{1}{24}\left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{120}$$

7-

$\iiint_E x^2 e^y \, dV$, where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$



E is the region below the parabolic cylinder $z=1-y^2$ and above the square $[-1,1]\times[-1,1]$ in the xy – plane.

$$\iiint_E x^2 e^y \, dV = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2 e^y \, dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} x^2 e^y (1-y^2)\,dy\,dx$$

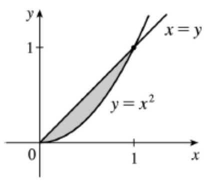
$$= \int_{-1}^{1} x^2\,dx \int_{-1}^{1} (e^y - y^2 e^y)\,dy$$

$$= \left[\frac{1}{3}x^3\right]_{-1}^{1}\left[e^y - (y^2-2y+2)e^y\right]_{-1}^{1}$$

$$= \frac{1}{3}(2)[e-e-e^{-1}+5e^{-1}] = \frac{8}{3e}$$

8-

$\iiint_E (x + 2y) \, dV$, where E is bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$, and $z = 0$



E is the solid above the region shown in the xy –plane and below the plane $z=x$. Thus,

$$\iiint_E (x+2y)\,dV = \int_0^1\int_{x^2}^{x}\int_0^{x} (x+2y)\,dz\,dy\,dx$$

$$= \int_0^1\int_{x^2}^{x} (x^2+2yx)\,dy\,dx = \int_0^1 \left[x^2 y + xy^2\right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (2x^3 - x^4 - x^5)\,dx = \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6\right]_0^1 = \frac{2}{15}$$