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CALCULUS FOR ENGINEERS
MATH 1110

Tutorial No. 14 Solution

Density, Mass, Moment and Center of Mass

Find the mass and center of mass of the lamina that occupies the region D and has the given density function

1-

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ -1 \leqslant y \leqslant 1\};\ \rho(x, y) = xy^2$$

$$m=\iint_D \rho(x,y)\,dA=\int_0^2\int_{-1}^1 xy^2\,dy\,dx=\int_0^2 x\,dx\int_{-1}^1 y^2\,dy=\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=2\cdot\frac{2}{3}=\frac{4}{3}\ ,$$

$$\bar{x}=\frac{1}{m}\iint_D x\rho(x,y)\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx=\frac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy=\frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=\frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3}=\frac{4}{3}\ ,$$

$$\bar{y}=\frac{1}{m}\iint_D y\rho(x,y)\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx=\frac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy=\frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1=\frac{3}{4}\cdot 2\cdot 0=0\ .$$

Hence, $(\bar{x},\bar{y})=\left(\frac{4}{3},0\right)$.

2-

$$D = \{(x, y) \mid 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\};\ \rho(x, y) = cxy$$

$$m=\iint_D \rho(x,y)\,dA=\int_0^a\int_0^b cxy\,dy\,dx=c\int_0^a x\,dx\int_0^b y\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{4}a^2b^2c\ ,$$

$$M_y=\iint_D x\rho(x,y)\,dA=\int_0^a\int_0^b cx^2y\,dy\,dx=c\int_0^a x^2\,dx\int_0^b y\,dy=c\left[\frac{1}{3}x^3\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{6}a^3b^2c\ ,\text{ and}$$

$$M_x=\iint_D y\rho(x,y)\,dA=\int_0^a\int_0^b cxy^2\,dy\,dx=c\int_0^a x\,dx\int_0^b y^2\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{3}y^3\right]_0^b=\frac{1}{6}a^2b^3c\ .$$

Hence, $(\bar{x},\bar{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{2}{3}a,\frac{2}{3}b\right)$.

3-

D is the triangular region with vertices (0, 0), (2, 1), (0, 3); $\rho(x, y) = x + y$

$$m=\int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx=\int_0^2\left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^2-\frac{1}{8}x^2\right]dx$$

$$=\int_0^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx=\left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2=6\,,$$

$$M_y=\int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx=\int_0^2\left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left(\frac{9}{2}x-\frac{9}{8}x^3\right)dx=\frac{9}{2}\text{ , and}$$

$$M_x=\int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx=\int_0^2\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left(9-\frac{9}{2}x\right)dx=9\text{ . Hence } m=6\,,$$

$$(\overline{x},\overline{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{3}{4},\frac{3}{2}\right)\,.$$

4-

D is the triangular region with vertices (0, 0), (1, 1), (4, 0); $\rho(x, y) = x$

$$m=\int_0^1\int_y^{4-3y}x\,dx\,dy=\int_0^1\left[\frac{1}{2}(4-3y)^2-\frac{1}{2}y^2\right]dy=\left[-\frac{1}{18}(4-3y)^3-\frac{1}{6}y^3\right]_0^1=\frac{10}{3}\,,$$

$$M_y=\int_0^1\int_y^{4-3y}x^2\,dx\,dy=\int_0^1\left[\frac{1}{3}(4-3y)^3-\frac{1}{3}y^3\right]dy=\left[-\frac{1}{36}(4-3y)^4-\frac{1}{12}y^4\right]_0^1=7\,,$$

$$M_x=\int_0^1\int_y^{4-3y}xy\,dx\,dy=\int_0^1\left[\frac{1}{2}y(4-3y)^2-\frac{1}{2}y^3\right]dy=\int_0^1(8y-12y^2+4y^3)\,dy=1\,.$$

Hence $m=\dfrac{10}{3}$, $(\overline{x},\overline{y})=(2.1,0.3)$.

5-

D is bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$; $\rho(x, y) = y$

$$m=\int_0^1\int_0^{e^x} y\,dy\,dx=\int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx=\frac{1}{2}\int_0^1 e^{2x}dx=\left.\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{4}(e^2-1),$$

$$M_y=\int_0^1\int_0^{e^x} xy\,dy\,dx=\frac{1}{2}\int_0^1 xe^{2x}dx=\frac{1}{2}\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{8}(e^2+1)\text{, and}$$

$$M_x=\int_0^1\int_0^{e^x} y^2\,dy\,dx=\int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x}dx=\frac{1}{3}\int_0^1 e^{3x}dx=\frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1=\frac{1}{9}(e^3-1).$$

Hence $m=\frac{1}{4}(e^2-1)$, $(\bar{x},\bar{y})=\left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right)=\left(\dfrac{e^2+1}{2(e^2-1)},\dfrac{4(e^3-1)}{9(e^2-1)}\right)$.

6-

A lamina occupies the part of the disk $x^2 + y^2 \leqslant 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

$$\rho(x,y)=ky=kr\sin\theta\ ,\ m=\int_0^{\pi/2}\int_0^1 kr^2\sin\theta\,dr\,d\theta=\frac{1}{3}k\int_0^{\pi/2}\sin\theta\,d\theta=\frac{1}{3}k[-\cos\theta]_0^{\pi/2}=\frac{1}{3}k,$$

$$M_y=\int_0^{\pi/2}\int_0^1 kr^3\sin\theta\cos\theta\,dr\,d\theta=\frac{1}{4}k\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta=\frac{1}{8}k[-\cos 2\theta]_0^{\pi/2}=\frac{1}{8}k,$$

$$M_x=\int_0^{\pi/2}\int_0^1 kr^3\sin^2\theta\,dr\,d\theta=\frac{1}{4}k\int_0^{\pi/2}\sin^2\theta\,d\theta=\frac{1}{8}k[\theta+\sin 2\theta]_0^{\pi/2}=\frac{\pi}{16}k.$$

Hence $(\bar{x},\bar{y})=\left(\frac{3}{8},\frac{3\pi}{16}\right)$.

7-

Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y)=k(x^2+y^2)$. Then

$$m = \int_0^a \int_0^{a-x} k\left(x^2+y^2\right) dy\,dx = k\int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3 \right] dx$$

$$= k\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4 \right]_0^a = \frac{1}{6}ka^4$$

By symmetry,

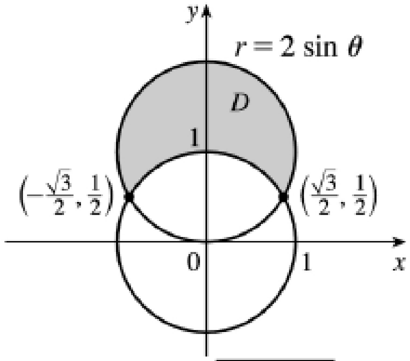
$$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a \left[\frac{1}{2}(a-x)^2x^2 + \frac{1}{4}(a-x)^4 \right] dx$$

$$= k\left[\frac{1}{6}a^2x^3 - \frac{1}{4}ax^4 + \frac{1}{10}x^5 - \frac{1}{20}(a-x)^5 \right]_0^a = \frac{1}{15}ka^5$$

Hence $(\bar{x},\bar{y}) = \left(\frac{2}{5}a, \frac{2}{5}a \right)$.

8-

A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.



$$\rho(x,y) = k/\sqrt{x^2+y^2} = k/r \;,$$

$$m = \int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} \frac{k}{r}\, r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1]\,d\theta$$

$$= k\left[-2\cos\theta - \theta\right]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \frac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \frac{1}{2}k\int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta)\,d\theta$$

$$= \frac{1}{2}k\left[-3\cos\theta + \frac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence $(\bar{x},\bar{y}) = \left(0, \dfrac{3\sqrt{3}}{2(3\sqrt{3}-\pi)}\right)$.

9-

Find the moments of inertia I_x, I_y, I_0 for the lamina of

D is bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$; $\rho(x, y) = y$

$$I_x = \iint_D y^2 \rho(x,y)\, dA = \int_0^1 \int_0^{e^x} y^2 \cdot y\, dy dx = \int_0^1 \left[\frac{1}{4} y^4 \right]_{y=0}^{y=e^x} dx = \frac{1}{4} \int_0^1 e^{4x} dx$$

$$= \frac{1}{4} \left[\frac{1}{4} e^{4x} \right]_0^1 = \frac{1}{16} (e^4 - 1) ,$$

$$I_y = \iint_D x^2 \rho(x,y) dA = \int_0^1 \int_0^{e^x} x^2 y dy dx = \int_0^1 x^2 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=e^x} dx = \frac{1}{2} \int_0^1 x^2 e^{2x} dx$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} x^2 - \frac{1}{2} x + \frac{1}{4} \right) e^{2x} \right]_0^1 \text{ [integrate by parts twice]}$$

$$= \frac{1}{8} (e^2 - 1) ,$$

and $I_0 = I_x + I_y = \frac{1}{16} (e^4 - 1) + \frac{1}{8} (e^2 - 1) = \frac{1}{16} (e^4 + 2e^2 - 3)$.

10-

Find the moments of inertia I_x, I_y, I_0 for the lamina of

D is bounded by the parabola $x = y^2$ and the line $y = x - 2$;
$\rho(x, y) = 3$

$$I_x = \int_{-1}^{2} \int_{y^2}^{y+2} 3y^2\, dx\, dy = \int_{-1}^{2} (3y^3 + 6y^2 - 3y^4)\, dy = \left[\frac{3}{4} y^4 + 2y^3 - \frac{3}{5} y^5 \right]_{-1}^{2} = \frac{189}{20} \ ,$$

$$I_y = \int_{-1}^{2} \int_{y^2}^{y+2} 3x^2\, dx\, dy = \int_{-1}^{2} \left[(y+2)^3 - y^6 \right] dy = \left[\frac{1}{4} (y+2)^4 - \frac{1}{7} y^7 \right]_{-1}^{2} = \frac{1269}{28} \ \text{, and } I_0 = I_x + I_y = \frac{1917}{35} \ .$$