Tutorial No. 15 Solution

Moments of Inertia

Find the moments of inertia for the lamina of

1- Find the moments of inertia for the lamina occupies the part of the disk $x^2 + y^2 \le 1$ in the first quadrant. If the density at any point is proportional to the square of its distance from the origin.

$$I_x = \int_0^{\pi/2}\int_0^1 (r^2 \sin^2\theta)(kr^2) r\,dr\,d\theta = \frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta - \sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k\,,$$

$$I_y = \int_0^{\pi/2}\int_0^1 (r^2 \cos^2\theta)(kr^2) r\,dr\,d\theta = \frac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta + \sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k\,, \text{ and}$$

$$I_0 = I_x + I_y = \frac{\pi}{12}k\,.$$

$$17.\ I_x = \int_{-1}^{2}\int_{y^2}^{y+2} 3y^2\,dx\,dy = \int_{-1}^{2}(3y^3 + 6y^2 - 3y^4)\,dy = \left[\frac{3}{4}y^4 + 2y^3 - \frac{3}{5}y^5\right]_{-1}^{2} = \frac{189}{20}\ ,$$

$$I_y = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2\,dx\,dy = \int_{-1}^{2}\left[(y+2)^3 - y^6\right]dy = \left[\frac{1}{4}(y+2)^4 - \frac{1}{7}y^7\right]_{-1}^{2} = \frac{1269}{28}\ , \text{ and } I_0 = I_x + I_y = \frac{1917}{35}\ .$$

2-

Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the x-axis or the y-axis?

$$I_x = \iint_D y^2 \rho(x,y)\, dA = \int_0^2 \int_0^2 y^2 (1+0.1x)\, dy\, dx = \int_0^2 (1+0.1x) \left[\frac{1}{3} y^3 \right]_{y=0}^{y=2} dx$$

$$= \frac{8}{3} \int_0^2 (1+0.1x)\, dx = \frac{8}{3} \left[x + 0.1 \cdot \frac{1}{2} x^2 \right]_0^2 = \frac{8}{3} (2.2) \approx 5.87$$

Similarly, the moment of inertia about the y –axis is given by

$$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_0^2 \int_0^2 x^2 (1+0.1x)\, dy\, dx = \int_0^2 x^2 (1+0.1x) [y]_{y=0}^{y=2} dx$$

$$= 2\int_0^2 (x^2 + 0.1x^3)\, dx = 2 \left[\frac{1}{3} x^3 + 0.1 \cdot \frac{1}{4} x^4 \right]_0^2 = 2 \left(\frac{8}{3} + 0.4 \right) \approx 6.13$$

Since

$I_y > I_x$, more force is required to rotate the fan blade about the y –axis.

3-

A lamina with constant density $\rho(x, y) = \rho$ occupies a square with vertices $(0, 0)$, $(a, 0)$, (a, a), and $(0, a)$. Find the moments of inertia I_x and I_y

$$I_x = \int_0^a \int_0^a \rho y^2 \, dx\, dy = \rho \int_0^a dx \int_0^a y^2 \, dy = \rho \, [x]_0^a \left[\frac{1}{3} y^3 \right]_0^a$$

$$= \rho a \left(\frac{1}{3} a^3 \right) = \frac{1}{3} \rho a^4 = I_y \text{ by symmetry}$$

4-

A lamina with constant density $\rho(x, y) = \rho$ occupies the region under the curve $y = \sin x$ from $x = 0$ to $x = \pi$. Find the moments of inertia I_x and I_y

$$m = \int_0^{\pi} \int_0^{\sin x} \rho \, dy\, dx = \rho \int_0^{\pi} \sin x \, dx = \rho \, [-\cos x]_0^{\pi} = 2\rho \ ,$$

$$I_x = \int_0^{\pi} \int_0^{\sin x} \rho y^2 \, dy\, dx = \frac{1}{3} \rho \int_0^{\pi} \sin^3 x \, dx = \frac{1}{3} \rho \int_0^{\pi} (1 - \cos^2 x) \sin x \, dx$$

$$= \frac{1}{3} \rho \left[-\cos\theta + \frac{1}{3} \cos^3\theta \right]_0^{\pi} = \frac{4}{9} \rho \ ,$$

$$I_y = \int_0^{\pi} \int_0^{\sin x} \rho x^2 \, dy\, dx = \rho \int_0^{\pi} x^2 \sin x \, dx$$

$$= \rho \left[-x^2 \cos x + 2x \sin x + 2\cos x \right]_0^{\pi}$$

$$= \rho (\pi^2 - 4) \ .$$