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CALCULUS FOR ENGINEERS
MATH 1110

Tutorial No. 9

Integral Basics

1- Evaluate the following integrals:

1-A-

$$\int_{-1}^{3} (3x^2 - 2x + 1)\, dx.$$

$\int (3x^2 - 2x + 1)\, dx = x^3 - x^2 + x$. (We omit the arbitrary constant in all such cases.) So $\int_{-1}^{3} (3x^2 - 2x + 1)\, dx = (x^3 - x^2 + x)\,\Big]_{-1}^{3} = (3^3 - 3^2 + 3) - [(-1)^3 - (-1)^2 + (-1)] = 21 - (-3) = 24.$

1-B-

$$\int_1^4 (x^2 - 4x + 2)\, dx$$

1-C-

$$\int_0^2 (2x - x^3)\, dx$$

1-D-

$$\int_0^{\pi/4} \cos x\, dx.$$

$\int \cos x\, dx = \sin x$. Hence, $\int_0^{\pi/4} \cos x\, dx = \sin x\,\Big]_0^{\pi/4} = \sin \frac{\pi}{4} - \sin 0 = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.$

1-E-

$$\int_0^{\pi/3} \sec^2 x\, dx.$$

$\int \sec^2 x\, dx = \tan x$. Hence, $\int_0^{\pi/3} \sec^2 x\, dx = \tan x\,\Big]_0^{\pi/3} = \tan \frac{\pi}{3} - \tan 0 = \sqrt{3} - 0 = \sqrt{3}.$

1-F-

$$\int_1^{16} x^{3/2}\, dx.$$

$\int x^{3/2}\, dx = \frac{2}{5}x^{5/2}$. Hence, $\int_1^{16} x^{3/2}\, dx = \frac{2}{5}x^{5/2}\Big]_1^{16} = \frac{2}{5}[(16)^{5/2} - (1)^{5/2}] = \frac{2}{5}[(\sqrt{16})^5 - (\sqrt{1})^5] = \frac{2}{5}(1024 - 1) = \frac{2}{5}(1023) = \frac{2046}{5}$.

1-G-

$$\int_4^5 \left(\frac{2}{\sqrt{x}} - x\right) dx.$$

$\int \left(\frac{2}{\sqrt{x}} - x\right) dx = \int (2x^{-1/2} - x)\, dx = 4x^{1/2} - \frac{1}{2}x^2$. Hence, $\int_4^5 \left(\frac{2}{\sqrt{x}} - x\right) dx = (4x^{1/2} - \frac{1}{2}x^2)\Big]_4^5 = [4\sqrt{5} - \frac{1}{2}(5)^2] - [4\sqrt{4} - \frac{1}{2}(4)^2] = (4\sqrt{5} - \frac{25}{2}) - (8 - 8) = 4\sqrt{5} - \frac{25}{2}$.

1-H-

$$\int_0^1 \sqrt{x^2 - 6x + 9}\, dx.$$

2- In the following problems calculate the area under the graph of the function f(x), above x-axix, and between the two indicated values a and b.

2-1

$$f(x) = \sin x, \quad a = \pi/6, \quad b = \pi/3.$$

$$A = \int_{\pi/6}^{\pi/3} \sin x\, dx = (-\cos x)\Big]_{\pi/6}^{\pi/3} = \left(-\cos\frac{\pi}{3}\right) - \left(-\cos\frac{\pi}{6}\right) = \left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}-1}{2}.$$

2-2

$$f(x) = x^2 + 4x, \quad a = 0, \quad b = 3.$$

$$A = \int_0^3 (x^2 + 4x)\, dx = \left(\tfrac{1}{3}x^3 + 2x^2\right)\Big]_0^3 = \left[\tfrac{1}{3}(3)^3 + 2(3)^2\right] = 9 + 18 = 27.$$

2-3

$$f(x) = 1/\sqrt[3]{x}, \quad a = 1, \quad b = 8.$$

$$A = \int_1^8 \frac{1}{\sqrt[3]{x}}\, dx = \tfrac{3}{2} x^{2/3} \Big]_1^8 = \tfrac{3}{2}(8^{2/3} - 1^{2/3}) = \tfrac{3}{2}(4-1) = \tfrac{9}{2}.$$

2-4

$$f(x) = \sqrt{4x+1}, \quad a = 0, \quad b = 2.$$

▮ $A = \int_0^2 \sqrt{4x+1}\, dx$. To find $\int \sqrt{4x+1}\, dx$, let $u = 4x+1$, $du = 4\, dx$. $\int \sqrt{4x+1}\, dx = \frac{1}{4} \int u^{1/2}\, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} = \frac{1}{6}(\sqrt{4x+1})^3$. So, $A = \frac{1}{6}(\sqrt{4x+1})^3 \Big]_0^2 = \frac{1}{6}[(\sqrt{9})^3 - (\sqrt{1})^3] = \frac{1}{6}(27-1) = \frac{13}{3}$.

2-5

$$f(x) = x^2 - 3x, \quad a = 3, \quad b = 5.$$

▮ $A = \int_3^5 (x^2 - 3x)\, dx = (\frac{1}{3}x^3 - \frac{3}{2}x^2) \Big]_3^5 = [\frac{1}{3}(5)^3 - \frac{3}{2}(5)^2] - [\frac{1}{3}(3)^3 - \frac{3}{2}(3)^2] = \frac{25}{6} + \frac{9}{2} = \frac{26}{3}$.

2-6

$$f(x) = \sin^2 x \cos x, \quad a = 0, \quad b = \pi/2.$$

▮ $A = \int_0^{\pi/2} \sin^2 x \cos x\, dx = \frac{1}{3} \sin^3 x \Big]_0^{\pi/2} = \frac{1}{3}[\sin^3 (\pi/2) - \sin^3 0] = \frac{1}{3}$.

3- Find the following definite integral

3-1

$$\int_{-1}^{1} \sqrt{3x^2 - 2x + 3}\, (3x-1)\, dx.$$

▮ To find $\int \sqrt{3x^2 - 2x + 3}\,(3x-1)\, dx$, let $u = 3x^2 - 2x + 3$, $du = (6x-2)\, dx = 2(3x-1)\, dx$. So, $\int \sqrt{3x^2 - 2x + 3}\, (3x-1)\, dx = \frac{1}{2} \int \sqrt{u}\, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} = \frac{1}{3}(3x^2 - 2x + 3)^{3/2}$. Hence, $\int_{-1}^{1} \sqrt{3x^2 - 2x + 3}(3x - 1)\, dx = \frac{1}{3}(3x^2 - 2x + 3)^{3/2} \Big]_{-1}^{1} = \frac{1}{3}[(3-2+3)^{3/2} - (3+2+3)^{3/2}] = \frac{1}{3}(8 - 16\sqrt{2}) = \frac{8}{3}(1 - 2\sqrt{2})$.

3-2

$$\int_{-1}^{2} \sqrt{x+2}\, x^2\, dx.$$

▮ Let $u = x + 2$, $x = u - 2$, $du = dx$. When $x = -1$, $u = 1$, and, when $x = 2$, $u = 4$. Then, by change of variables, $\int_{-1}^{2} \sqrt{x+2}\, x^2\, dx = \int_1^4 \sqrt{u}(u-2)^2\, du = \int_1^4 \sqrt{u}(u^2 - 4u + 4)\, du = \int_1^4 (u^{5/2} - 4u^{3/2} + 4u^{1/2})\, du = [\frac{2}{7}u^{7/2} - 4(\frac{2}{5})u^{5/2} + 4(\frac{2}{3})u^{3/2}] \Big]_1^4 = 2u^{3/2}(\frac{1}{7}u^2 - \frac{4}{5}u + \frac{4}{3}) \Big]_1^4 = 2[8(\frac{16}{7} - \frac{16}{5} + \frac{4}{3}) - (\frac{1}{7} - \frac{4}{5} + \frac{4}{3})] = 2(\frac{127}{7} - \frac{124}{5} + \frac{28}{3}) = \frac{562}{105}$.

3-3

$$\int_2^5 \sqrt{x^3-4}\, x^5\, dx.$$

Let $u = x^3 - 4$, $x^3 = u + 4$, $du = 3x^2\,dx$. When $x = 2$, $u = 4$, and, when $x = 5$, $u = 121$. Then $\int_2^5 \sqrt{x^3-4}\,x^5\,dx = \frac{1}{3}\int_4^{121} \sqrt{u}(u+4)\,du = \frac{1}{3}\int_4^{121} (u^{3/2} + 4u^{1/2})\,du = \frac{1}{3}(\frac{2}{5}u^{5/2} + \frac{8}{3}u^{3/2})\,]_4^{121} = \frac{2}{45}u^{3/2}(3u+20)\,]_4^{121}$ $= \frac{2}{45}[(11)^3(383) - 8(32)] = \dfrac{113{,}226}{5}$.

3-4

$$\int_3^{15} \sqrt[3]{x^2-9}\, x^3\, dx.$$

Let $u = x^2 - 9$, $x^2 = u + 9$, $du = 2x\,dx$. Then $\int_3^{15} \sqrt[3]{x^2-9}\,x^3\,dx = \frac{1}{2}\int_0^{216} u^{1/3}(u+9)\,du = \frac{1}{2}\int_0^{216} (u^{4/3} + 9u^{1/3})\,du = \frac{1}{2}(\frac{3}{7}u^{7/3} + 9\cdot\frac{3}{4}u^{4/3})\,]_0^{216} = \frac{3}{56}u^{4/3}(4u+63)\,]_0^{216} = \frac{3}{56}\{(6)^4[4(216)+63] - 0\} = \dfrac{349{,}522}{7}$.

3-5

$$\int_0^1 \frac{x}{(2x^2+1)^3}\, dx.$$

Let $u = 2x^2 + 1$, $du = 4x\,dx$. Then $\displaystyle\int_0^1 \frac{x}{(2x^2+1)^3}\,dx = \frac{1}{4}\int_1^3 u^{-3}\,du = -\frac{1}{8}u^{-2}\,]_1^3 = -\frac{1}{8}(\frac{1}{9} - 1) = \frac{1}{9}$.

3-6

$$\int_0^8 \frac{x}{(x+1)^{3/2}}\, dx.$$

Let $u = x + 1$, $x = u - 1$, $du = dx$. Then $\displaystyle\int_0^8 \frac{x}{(x+1)^{3/2}}\,dx = \int_1^9 \frac{u-1}{u^{3/2}}\,du = \int_1^9 (u^{-1/2} - u^{-3/2})\,du =$ $(2u^{1/2} + 2u^{-1/2})\,]_1^9 = 2[(3 + \frac{1}{3}) - (1+1)] = \frac{8}{3}$.

3-7

$$\int_1^2 \frac{x^7 - 2x + 1}{4x^3}\, dx.$$

$\displaystyle\int_1^2 \frac{x^7-2x+1}{4x^3}\,dx = \frac{1}{4}\int_1^2 (x^4 - 2x^{-2} + x^{-3})\,dx = \frac{1}{4}(\frac{1}{5}x^5 + 2x^{-1} - \frac{1}{2}x^{-2})\,]_1^2 = \frac{1}{4}[(\frac{32}{5} + 1 - \frac{1}{8}) - (\frac{1}{5} + 2 - \frac{1}{2})]$ $= \frac{1}{4}(\frac{31}{5} - 1 + \frac{3}{8}) = \frac{223}{160}$.