Faculty of Engineering
Mechanical Engineering Department

Linear Algebra and Vector Analysis
MATH 1120
Lecture 10

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- Cramer’s Rule

Linear Equations and Determinants

- The solutions of linear equations can sometimes be expressed using determinants.
  - To illustrate, let's solve the following pair of linear equations for the variable *x*.

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

Linear Equations and Determinants

- To eliminate the variable *y*, we multiply the first equation by *d* and the second by *b*, and subtract.

$$
\begin{array}{l}
adx + bdy = rd \\
bcx + bdy = bs \\
\hline
adx - bcx = rd - bs
\end{array}
$$

Linear Equations and Determinants

- Factoring the left-hand side, we get:

$$(ad - bc)x = rd - bs$$

  – Assuming that $ad - bc \neq 0$, we can now solve this equation for x:

$$x = \frac{rd - bs}{ad - bc}$$

  – Similarly, we find:

$$y = \frac{as - cr}{ad - bc}$$

Linear Equations and Determinants

- The numerator and denominator of the fractions for *x* and *y* are determinants of 2 x 2 matrices.
  - So, we can express the solution of the system using determinants as follows.

Cramer's Rule for Systems in Two Variables

- The linear system $\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$

  has the solution

$$x = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

  provided

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

Cramer's Rule

- Using the notation

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad D_x = \begin{bmatrix} r & b \\ s & d \end{bmatrix} \quad D_y = \begin{bmatrix} a & r \\ c & s \end{bmatrix}$$

the solution of the system can be written as:

$$x = \frac{|D_x|}{|D|} \quad \text{and} \quad y = \frac{|D_y|}{|D|}$$

E.g. 6—Cramer's Rule for a System with Two Variables

- Use Cramer's Rule to solve the system.

$$\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$$

E.g. 6—Cramer's Rule for a System with Two Variables

- For this system, we have:

$$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 \cdot 8 - 6 \cdot 1 = 10$$

$$|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)8 - 6 \cdot 2 = -20$$

$$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)1 = 5$$

E.g. 6—Cramer's Rule for a System with Two Variables

- The solution is:

$$x = \frac{|D_x|}{|D|} = \frac{-20}{10} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{5}{10} = \frac{1}{2}$$

Cramer's Rule

- Cramer's Rule can be extended to apply to any system of n linear equations in n variables in which the determinant of the coefficient matrix is not zero.

Cramer's Rule

- As we saw in the preceding section, any such system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer's Rule

- By analogy with our derivation of Cramer's Rule in the case of two equations in two unknowns, we let:
  - D be the coefficient matrix in this system.
  - D_{x_i} be the matrix obtained by replacing the ith column of D by the numbers $b_1, b_2, \ldots, b_n$ that appear to the right of the equal sign.

- The solution of the system is then given by the following rule.

Cramer's Rule

- Suppose a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$, and $|D| \neq 0$.
  - Then, its solutions are:

$$x_1 = \frac{|D_{x_1}|}{|D|},\ x_2 = \frac{|D_{x_2}|}{|D|},\ \ldots,\ x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the n x 1 matrix B.

E.g. 7—Cramer's Rule for a System of Three Variables

- Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - 3y + 4z = 1 \\ x \qquad\quad + 6z = 0 \\ 3x - 2y \qquad\quad = 5 \end{cases}$$

  - First, we evaluate the determinants that appear in Cramer's Rule.

E.g. 7—Cramer's Rule for a System of Three Variables

$\|D\| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38$	$\|D_x\| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$
$\|D_y\| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22$	$\|D_z\| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$

– Note that D is the coefficient matrix and that D_x, D_y, and D_z are obtained by replacing the first, second, and third columns of D by the constant terms.

E.g. 7—Cramer's Rule for a System of Three Variables

- Now, we use Cramer's Rule to get the solution:

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19}$$

$$y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19}$$

$$z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

Cramer's Rule

- Solving the system in Example 7 using Gaussian elimination would involve matrices whose elements are fractions with fairly large denominators.

  – Thus, in cases like Examples 6 and 7, Cramer's Rule gives us an efficient way to solve systems of linear equations.

Limitations of Cramer's Rule

- However, in systems with more than three equations, evaluating the various determinants involved is usually a long and tedious task.
  - This is unless you are using a graphing calculator.

Limitations of Cramer's Rule

- Moreover, the rule doesn't apply if $|D| = 0$ or if D is not a square matrix.

  – So, Cramer's Rule is a useful alternative to Gaussian elimination—but only in some situations.