Faculty of Engineering
Mechanical Engineering Department

Linear Algebra and Vector Analysis
MATH 1120

: Instructor
Dr. O. Philips Agboola
pagboola@ksu.edu.sa
Office: F-092

Anton/Busby CONTEMPORARY LINEAR ALGEBRA

Chapter 2, Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

The study of systems of linear equations and their solutions is one of the major topics in linear algebra. In this introductory section we will discuss some ways in which systems of linear equations arise, what it means to solve them, and how their solutions can be interpreted geometrically. Our focus here will be on general ideas, and in the next section we will discuss computational methods for finding solutions.

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Chapter 2, Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

Linear systems in two unknowns arise in connection with intersections of lines in R^2. For example, consider the linear system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

The graphs of these equations are lines in the xy-plane, so each solution (x, y) of this system corresponds to a point of intersection of these lines. Thus, there are three possibilities (Figure 2.1.1):

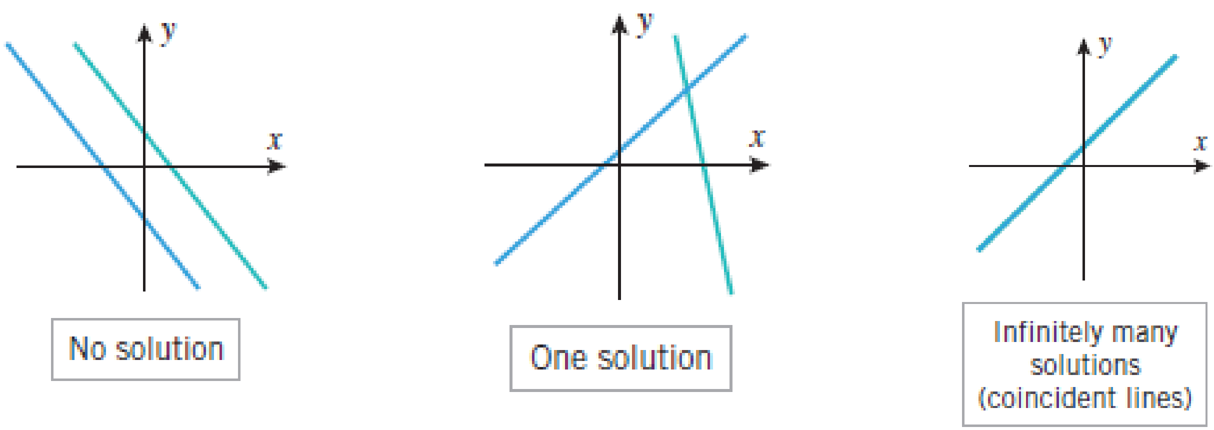

Figure 2.1.1

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Chapter 2, Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

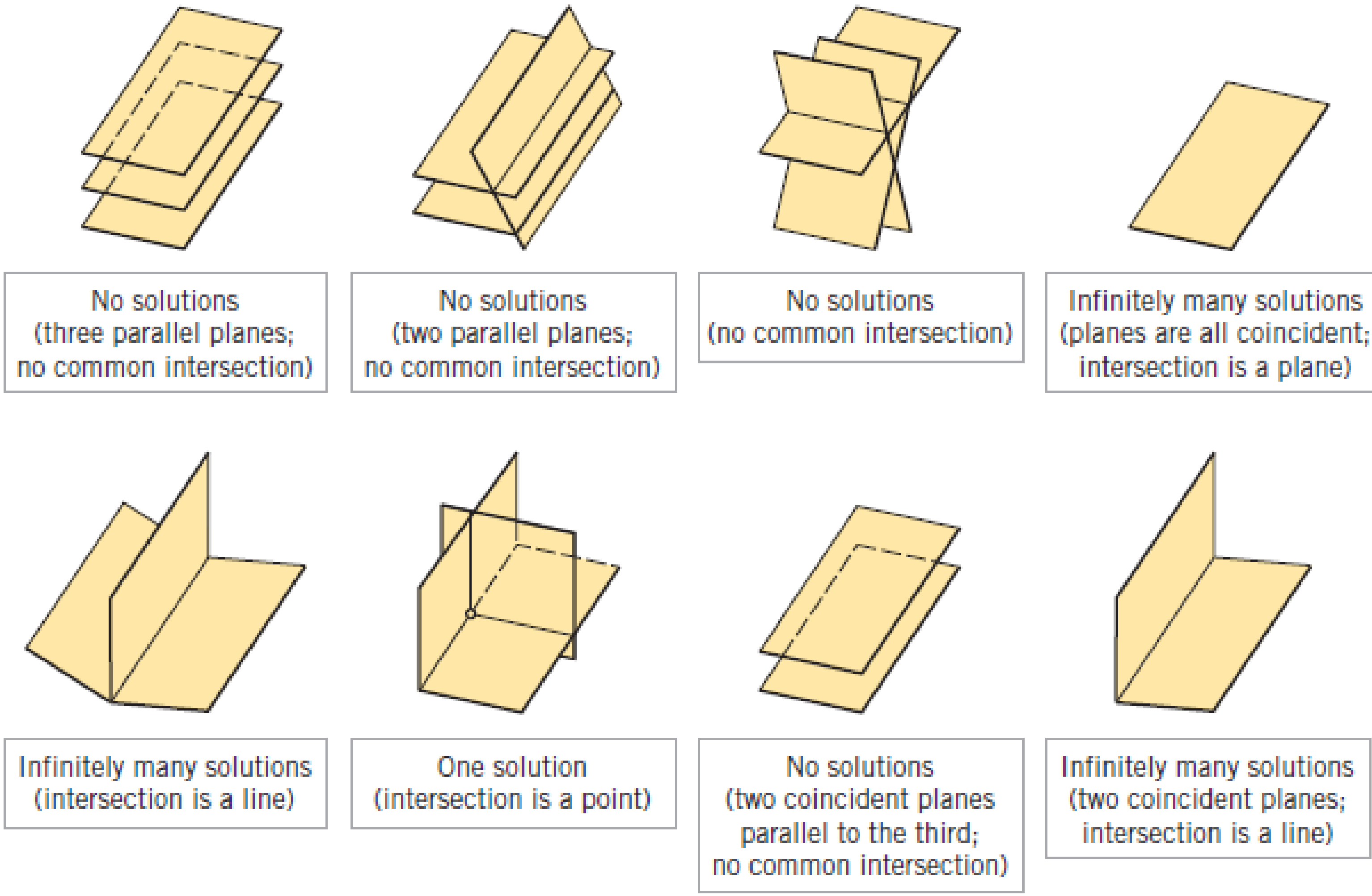

Figure 2.1.2

The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$

$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

Theorem 2.1.1 *Every system of linear equations has zero, one, or infinitely many solutions; there are no other possibilities.*

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution $x = \frac{7}{3}$, $y = \frac{4}{3}$. Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines. ■

EXAMPLE 2
A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different y-intercepts. ■

EXAMPLE 3
A Linear System with No Solutions

EXAMPLE 4
A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and y and hence can be eliminated. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \tag{7}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. The most convenient way to describe the solution set in this case is to express (7) parametrically. We can do this by letting $y = t$ and solving for x in terms of t, or by letting $x = t$ and solving for y in terms of t. The first approach yields the following parametric equations:

$$x = \tfrac{1}{4} + \tfrac{1}{2}t, \ \ y = t$$

We can now obtain specific solutions by substituting numerical values for the parameter. For example, $t = 0$ yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$ yields the solution $\left(\frac{3}{4}, 1\right)$, and $t = -1$ yields the solution $\left(-\frac{1}{4}, -1\right)$. You can confirm that these are solutions by substituting the coordinates into the given equation. ■

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Solve the linear system

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - y + 2z = 5 \tag{8}$$

automatically satisfy all three equations. Using the method of Example 7 in Section 1.3, we can express the solution set parametrically as

$$x = 5 + t_1 - 2t_2,\ \ y = t_1,\ \ z = t_2$$

Specific solutions can be obtained by choosing numerical values for the parameters. ■

EXAMPLE 5
A Linear System with Infinitely Many Solutions

AUGMENTED MATRICES AND ELEMENTARY ROW OPERATIONS

keeping track of the location of the +'s, the x's, and the ='s in the linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

EXAMPLE 6

Using Elementary Row Operations and Augmented Matrices

In the left column we solve a linear system by operating on the equations, and in the right column we solve it by operating on the rows of the augmented matrix.

System

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add −2 times the first equation to the second to obtain

Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −2 times the first row to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −3 times the first equation to the third to obtain

Add −3 times the first row to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add −3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add −3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by −2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by −2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add −1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add −1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad\quad &= 1 \\ y \quad\; &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution

$$x = 1, \quad y = 2, \quad z = 3$$

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