Faculty of Engineering
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Linear Algebra and Vector Analysis
MATH 1120

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The Algebra of Matrices

- Thus far, we've used matrices simply for notational convenience when solving linear systems.

- Matrices have many other uses in mathematics and the sciences.

  – For most of these applications, a knowledge of matrix algebra is essential.

The Algebra of Matrices

- Like numbers, matrices can be added, subtracted, multiplied, and divided.
  - In this section, we learn how to perform these algebraic operations on matrices.

Equality of Matrices

- Two matrices are equal if they have the same entries in the same positions.

Equal Matrices	Unequal Matrices
$\begin{bmatrix} \sqrt{4} & 2^2 & e^0 \\ 0.5 & 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ \frac{1}{2} & \frac{2}{2} & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Equality of Matrices

- The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if:
  - They have the same dimension m x n.
  - Corresponding entries are equal.
    That is,

$$a_{ij} = b_{ij}$$

    for $i = 1, 2, . . . , m$
    and $j = 1, 2, . . . , n$.

E.g.1—Equal Matrices

- Find *a*, *b*, *c*, and *d*, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$

  - Since the two matrices are equal, corresponding entries must be the same.
  - So, we must have:

$$a = 1,\ b = 3,\ c = 5,\ d = 2$$

Addition and Subtraction of Matrices

- Two matrices can be added or subtracted if they have the same dimension.
  - Otherwise, their sum or difference is undefined.
  - We add or subtract the matrices by adding or subtracting corresponding entries.

Scalar Multiplication of Matrices

- To multiply a matrix by a number, we multiply every element of the matrix by that number.
  - This is called the scalar product

Sum, Difference, and Scalar Product of Matrices

- Let:
  - $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension m x n.
  - c be any real number.

Sum of Matrices

- The sum $A + B$ is the m x n matrix obtained by adding corresponding entries of A and B.

$$A + B = [a_{ij} + b_{ij}]$$

Difference of Matrices

- The difference $A - B$ is the m x n matrix obtained by subtracting corresponding entries of A and B.

$$A - B = [a_{ij} - b_{ij}]$$

Scalar Product of Matrices

- The scalar product cA is the m x n matrix obtained by multiplying each entry of A by c.

$$cA = [ca_{ij}]$$

E.g. 2—Performing Algebraic Operations in Matrices

- Let:

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

  – Carry out each indicated operation, or explain why it cannot be performed.
    (a) $A + B$ (b) $C - D$ (c) $C + A$ (d) $5A$

E.g. 2—Performing Algebraic Operations in Matrices

- (a)

$$A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$$

- (b)

$$C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$$

E.g. 2—Performing Algebraic Operations in Matrices

- (c) *C* + *A* is undefined because we can't add matrices of different dimensions.

- (d)

$$5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$$

Addition and Scalar Multiplication of Matrices

- The following properties follow from:
  - The definitions of matrix addition and scalar multiplication.
  - The corresponding properties of real numbers.

Properties of Addition and Scalar Multiplication

- Let A, B, and C be m x n matrices and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = (cd)A$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$ $c(A + B) = cA + cB$	Distributive Properties of Scalar Multiplication

E.g. 3—Solving a Matrix Equation

- Solve the matrix equation

$$2X - A = B$$

for the unknown matrix *X*, where:

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

E.g. 3—Solving a Matrix Equation

- We use the properties of matrices to solve for X.

$$2X - A = B$$

$$2X = B + A$$

$$X = \tfrac{1}{2}(B + A)$$

E.g. 3—Solving a Matrix Equation

- Thus,

$$X = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 2 \\ -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

- Multiplication of Matrices

Multiplication of Matrices

- Multiplying two matrices is more difficult to describe than other matrix operations.
  - In later examples, we will see why taking the matrix product involves a rather complex procedure—which we now describe.

Multiplication of Matrices

- First, the product AB (or $A \cdot B$) of two matrices A and B is defined only when:
  - The number of columns in A is equal to the number of rows in B.

Multiplication of Matrices

- This means that, if we write their dimensions side by side, the two inner numbers must match:

Matrices	*A*	*B*
Dimensions	m x n	n x k

Columns in *A*   Rows in *B*

-

Multiplication of Matrices

- If the dimensions of *A* and *B* match in this fashion, then the product *AB* is a matrix of dimension *m* x *k*.
  - Before describing the procedure for obtaining the elements of *AB*, we define the inner product of a row of *A* and a column of *B*.

Inner Product

- For example, taking the inner product of $[2 \;\; -1 \;\; 0 \;\; 4]$ and $\begin{bmatrix} 5 \\ 4 \\ -3 \\ \frac{1}{2} \end{bmatrix}$ gives:

$$2 \cdot 5 + (-1) \cdot 4 + 0 \cdot (-3) + 4 \cdot \tfrac{1}{2} = 8$$

Matrix Multiplication

- If $A = [a_{ij}]$ is an m x n matrix and $B = [b_{ij}]$ an n x k matrix, their product is the m x k matrix
- $C = [c_{ij}]$
  where c_{ij} is the inner product of the *i*th row of A and the *j*th column of B.

  – We write the product as $C = AB$

Multiplication of Matrices

- This definition of matrix product says that each entry in the matrix *AB* is obtained from a row of *A* and a column of *B*, as follows.

Multiplication of Matrices

- The entry c_{ij} in the *i*th row and *j*th column of the matrix *AB* is obtained by:
  - Multiplying the entries in the *i*th row of *A* with the corresponding entries in the *j*th column of *B.*
  - Adding the results.

$$\begin{bmatrix} \square & \square & \square \end{bmatrix} \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} c_{ij} \end{bmatrix}$$

E.g. 4—Multiplying Matrices

- Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

Calculate, if possible, the products *AB* and *BA*.

E.g. 4—Multiplying Matrices

- Since A has dimension 2 x 2 and B has dimension 2 x 3, the product AB is defined and has dimension 2 x 3.
  - We can thus write:

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

  - The question marks must be filled in using the rule defining the product of two matrices.

E.g. 4—Multiplying Matrices

- If we define $C = AB = [c_{ij}]$, the entry c_{11} is the inner product of the first row of A and the first column of B:

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \quad 1 \cdot (-1) + 3 \cdot 0 = -1$$

  – Similarly, we calculate the remaining entries of the product as follows.

E.g. 4—Multiplying Matrices

Entry	Inner Product of	Value	Product Matrix
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 5 + 3 \cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 2 + 3 \cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$

E.g. 4—Multiplying Matrices

Entry	Inner Product of	Value	Product Matrix
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot(-1) + 0\cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot 5 + 0\cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot 2 + 0\cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

E.g. 4—Multiplying Matrices

- Thus, we have:

$$AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$$

  – However, the product *BA* is not defined—because the dimensions of *B* and *A* are 2 x 3 and 2 x 2.

  – The inner two numbers are not the same.

  – So, the rows and columns won't match up when we try to calculate the product.

- Properties of
  Matrix Multiplication

Properties of Matrix Multiplication

- Although matrix multiplication is not commutative, it does obey the Associative and Distributive Properties.

Properties of Matrix Multiplication

- Let A, B, and C be matrices for which the following products are defined.
  - Then,

$A(BC) = (AB)C$	Associative Property
$A(B + C) = AB + AC$ $(B + C)A = BA + CA$	Distributive Property

Properties of Matrix Multiplication

- The next example shows that, even when both *AB* and *BA* are defined, they aren't necessarily equal.
  - This result proves that matrix multiplication is not commutative.

E.g. 5—Matrix Multiplication Is Not Commutative

- Let

$$A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$$

- Calculate the products *AB* and *BA*.
  - Since both matrices *A* and *B* have dimension 2 x 2, both products *AB* and *BA* are defined, and each product is also a 2 x 2 matrix.

E.g. 5—Matrix Multiplication Is Not Commutative

$$AB = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 1 + 7 \cdot 9 & 5 \cdot 2 + 7 \cdot (-1) \\ (-3) \cdot 1 + 0 \cdot 9 & (-3) \cdot 2 + 0 \cdot (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix}$$

E.g. 5—Matrix Multiplication Is Not Commutative

$$BA = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 5 + 2 \cdot (-3) & 1 \cdot 7 + 2 \cdot 0 \\ 9 \cdot 5 + (-1) \cdot (-3) & 9 \cdot 7 + (-1) \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix}$$

- This shows that, in general, $AB \neq BA$.
- In fact, in this example, *AB* and *BA* don't even have an entry in common.

- Applications of Matrix Multiplication

Applications of Matrix Multiplication

- We now consider some applied examples that give some indication of why mathematicians chose to define the matrix product in such an apparently bizarre fashion.
  - The next example shows how our definition of matrix product allows us to express a system of linear equations as a single matrix equation.

E.g. 6—Writing a Linear System as a Matrix Equation

- Show that this matrix equation is equivalent to the system of equations
in Example 2 of Section 7.1.

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

E.g. 6—Writing a Linear System as a Matrix Equation

- If we perform matrix multiplication on the left side of the equation, we get:

$$\begin{bmatrix} x - y + 3z \\ x + 2y - 2z \\ 3x - y + 5z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

E.g. 6—Writing a Linear System as a Matrix Equation

- Since two matrices are equal only if their corresponding entries are equal, we equate entries to get:

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

  - This is exactly the system of equations in Example 2 of Section 7.1.