Faculty of Engineering
Mechanical Engineering Department

Linear Algebra and Vector Analysis
MATH 1120

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- Inverses of Matrices and Matrix Equations

Introduction

- In the preceding section, we saw that, when the dimensions are appropriate, matrices can be added, subtracted, and multiplied.

- Here, we investigate division of matrices.
  - With this operation, we can solve equations that involve matrices.

- The Inverse of a Matrix

Identity Matrices

- First, we define identity matrices.
  - These play the same role for matrix multiplication as the number 1 does for ordinary multiplication of numbers.
  - That is,

$$1 \cdot a = a \cdot 1 = a$$

for all numbers *a*.

Identity Matrices

- In the following definition, the term main diagonal refers to the entries of a square matrix whose row and column numbers are the same.
  - These entries stretch diagonally down the matrix—from top left to bottom right.

Identity Matrix—Definition

- The identity matrix I_n is the *n* x *n* matrix for which:
  - Each main diagonal entry is a 1.
  - All other entries are 0.

Identity Matrices

- Thus, the

  2 x 2, 3 x 3, and 4 x 4

  identity matrices are:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity Matrices

- Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \quad \text{and} \quad I_n \cdot B = B$$

whenever these products are defined.

E.g. 1—Identity Matrices

- The following matrix products show how:
  - Multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

E.g. 1—Identity Matrices

- $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$

Inverse of a Matrix

- If A and B are n x n matrices, and if $AB = BA = I_n$, we say that B is the inverse of A, and we write $B = A^{-1}$.
  - The concept of the inverse of a matrix is analogous to that of the reciprocal of a real number.

Inverse of a Matrix—Definition

- Let A be a square n x n matrix.

- If there exists an n x n matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the inverse of A.

E.g. 2—Verifying that a Matrix Is an Inverse

- Verify that *B* is the inverse of *A*, where:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

  - We perform the matrix multiplications to show that $AB = I$ and $BA = I$.

E.g. 2—Verifying that a Matrix Is an Inverse

- $$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 3 + 1(-5) & 2(-1) + 1\cdot 2 \\ 5\cdot 3 + 3(-5) & 5(-1) + 3\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

E.g. 2—Verifying that a Matrix Is an Inverse

- $$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$
  $$= \begin{bmatrix} 3\cdot 2 + (-1)5 & 3\cdot 1 + (-1)3 \\ (-5)2 + 2\cdot 5 & (-5)1 + 2\cdot 3 \end{bmatrix}$$
  $$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Finding the Inverse of a 2 x 2 Matrix

Finding the Inverse of a 2 x 2 Matrix

- The following rule provides a simple way for finding the inverse of a 2 x 2 matrix, when it exists.
  - For larger matrices, there's a more general procedure for finding inverses—which we consider later in this section.

Inverse of a 2 x 2 Matrix

- If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- If $ad - bc = 0$, then A has no inverse.

E.g. 3—Finding the Inverse of a 2 x 2 Matrix

- Let A be the matrix

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

- Find A^{-1} and verify that

$$AA^{-1} = A^{-1}A = I_2$$

E.g. 3—Finding the Inverse of a 2 x 2 Matrix

- Using the rule for the inverse of a 2 x 2 matrix, we get:

$$A^{-1} = \frac{1}{4 \cdot 3 - 5 \cdot 2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

  – To verify that this is indeed the inverse of *A*, we calculate AA^{-1} and $A^{-1}A$.

E.g. 3—Finding the Inverse of a 2 x 2 Matrix

- $$AA^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot \frac{3}{2} + 5(-1) & 4(-\frac{5}{2}) + 5 \cdot 2 \\ 2 \cdot \frac{3}{2} + 3(-1) & 2(-\frac{5}{2}) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} \cdot 4 + (-\frac{5}{2})2 & \frac{3}{2} \cdot 5 + (-\frac{5}{2})3 \\ (-1)4 + 2 \cdot 2 & (-1)5 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

- The quantity $ad - bc$ that appears in the rule for calculating the inverse of a 2 x 2 matrix is called the determinant of the matrix.
  - If the determinant is 0, the matrix does not have an inverse (since we cannot divide by 0).

- Finding the Inverse of an n x n Matrix

Finding the Inverse of an n x n Matrix

- For 3 x 3 and larger square matrices, the following technique provides the most efficient way to calculate their inverses.

Finding the Inverse of an n x n Matrix

- If A is an n x n matrix, we first construct the n x $2n$ matrix that has the entries of A on the left and of the identity matrix I_n on the right:

$$\left[\begin{array}{cccc:cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

Finding the Inverse of an n x n Matrix

- We then use the elementary row operations on this new large matrix to change the left side into the identity matrix.
  - This means that we are changing the large matrix to reduced row-echelon form.

$$\left[\begin{array}{cccc:cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

Finding the Inverse of an n x n Matrix

- The right side is transformed automatically into A^{-1}.
  - We omit the proof of this fact.

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

E.g. 4—Finding the Inverse of a 3 x 3 Matrix

- Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Verify that $AA^{-1} = A^{-1}A = I_3$.

E.g. 4—Inverse of a 3 x 3 Matrix

Example (a)

- We begin with the 3 x 6 matrix whose left half is *A* and whose right half is the identity matrix.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

  – We then transform the left half of this new matrix into the identity matrix—by performing the following sequence of elementary row operations on the entire new matrix.

Example (a)

E.g. 4—Inverse of a 3 x 3 Matrix

$$\xrightarrow[R_3+3R_1 \to R_3]{R_2-2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

Example (a)

E.g. 4—Inverse of a 3 x 3 Matrix

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$$\xrightarrow{R_1+2R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2-2R_3\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

E.g. 4—Inverse of a 3 x 3 Matrix

- We have now transformed the left half of this matrix into an identity matrix.
  - This means we've put the entire matrix in reduced row-echelon form.

Example (a)

E.g. 4—Inverse of a 3 x 3 Matrix

- Note that, to do this in as systematic a fashion as possible, we first changed the elements below the main diagonal to zeros—just as we would if we were using Gaussian elimination.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

E.g. 4—Inverse of a 3 x 3 Matrix

- Then, we changed each main diagonal element to a 1 by multiplying by the appropriate constant(s).

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

E.g. 4—Inverse of a 3 x 3 Matrix

- Finally, we completed the process by changing the remaining entries on the left side to zeros.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

E.g. 4—Inverse of a 3 x 3 Matrix

- The right half is now A^{-1}.

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

E.g. 4—Inverse of a 3 x 3 Matrix

Example (b)

- We calculate AA^{-1} and $A^{-1}A$, and verify that both products give the identity matrix I_3.

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E.g. 5—Matrix that Does Not Have an Inverse

- Find the inverse of the matrix.

$$\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$

E.g. 5—Matrix that Does Not Have an Inverse

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$$\left[\begin{array}{ccc|ccc} 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

E.g. 5—Matrix that Does Not Have an Inverse

- $$\xrightarrow[R_3-R_1\to R_3]{R_2-2R_1\to R_2}\left[\begin{array}{rrr|rrr} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & -7 & -21 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2}\left[\begin{array}{rrr|rrr} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

E.g. 5—Matrix that Does Not Have an Inverse

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

– At this point, we would like to change the 0 in the (3, 3) position of this matrix to a 1, without changing the zeros in the (3, 1) and (3, 2) positions.

E.g. 5—Matrix that Does Not Have an Inverse

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

– However, there is no way to accomplish this.

– No matter what multiple of rows 1 and/or 2 we add to row 3, we can’t change the third zero in row 3 without changing the first or second zero as well.

E.g. 5—Matrix that Does Not Have an Inverse

$$\xrightarrow[R_1-2R_2\to R_1]{R_3+R_2\to R3}\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

– Thus, we cannot change the left half to the identity matrix.

– So, the original matrix doesn't have an inverse.

Matrix that Does Not Have an Inverse

- If we encounter a row of zeros on the left when trying to find an inverse—as in Example 5—then the original matrix does not have an inverse.