Faculty of Engineering
Mechanical Engineering Department

Linear Algebra and Vector Analysis
MATH 1120

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- Determinants and Cramer’s Rule

Determinants

- If a matrix is square (that is, if it has the same number of rows as columns), then we can assign to it a number called its determinant.
  - Determinants can be used to solve systems of linear equations--as we will see later in the section.
  - They are also useful in determining whether a matrix has an inverse.

- Determinant of a 2 x 2 Matrix

Determinant of 1 x 1 matrix

- We denote the determinant of a square matrix A by the symbol det(A) or | A |.

- We first define det(A) for the simplest cases.
  - If $A = [a]$ is a 1 x 1 matrix, then det(A) = a.

Determinant of a 2 x 2 Matrix

- The determinant of the 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

E.g. 1—Determinant of a 2 x 2 Matrix

- Evaluate | *A* | for $A = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

$$\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix} = 6 \cdot 3 - (-3)2 = 18 - (-6) = 24$$

- Determinant of an n x n Matrix

Determinant of an n x n Matrix

- To define the concept of determinant for an arbitrary $n \times n$ matrix, we need the following terminology.

Determinant of an n x n Matrix

- Let A be an n x n matrix.
  - The minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.
  - The cofactor A_{ij} of the element a_{ij} is:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Determinant of an n x n Matrix

- For example, A is the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$
  - The minor M_{12} is the determinant of the matrix obtained by deleting the first row and second column from A.

$$M_{12} = \begin{vmatrix} 2 & 3 & 1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} = 0(6) - 4(-2) = 8$$

  - So, the cofactor $A_{12} = (-1)^{1+2} M_{12} = -8$

Determinant of an n x n Matrix

- Similarly,
  - $$M_{33} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 0 = 4$$
  - So, $A_{33} = (-1)^{3+3} M_{33} = 4$

Determinant of an n x n Matrix

- Note that the cofactor of a_{ij} is simply the minor of a_{ij} multiplied by either 1 or –1, depending on whether $i + j$ is even or odd.
  - Thus, in a 3 x 3 matrix, we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Determinant of a Square Matrix

- We are now ready to define the determinant of any square matrix.

Determinant of a Square Matrix

- If A is an n x n matrix, the determinant of A is obtained by multiplying each element of the first row by its cofactor, and then adding the results.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

E.g. 2—Determinant of a 3 x 3 Matrix

- Evaluate the determinant of the matrix.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

E.g. 2—Determinant of a 3 x 3 Matrix

- $$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$$= 2(2 \cdot 6 - 4 \cdot 5) - 3[0 \cdot 6 - 4(-2)] - [0 \cdot 5 - 2(-2)]$$

$$= -16 - 24 - 4$$

$$= -44$$

Expanding the Determinant

- In our definition of the determinant, we used the cofactors of elements in the first row only.
  - This is called expanding the determinant by the first row.
  - In fact, we can expand the determinant by any row or column in the same way, and obtain the same result in each case.
  - We won’t prove this, though.

E.g. 3—Expanding Determinant about Row and Column

- Let *A* be the matrix of Example 2.
- Evaluate the determinant of *A* by expanding

(a) by the second row

(b) by the third column

  - Verify that each expansion gives the same value.

Example (a)

E.g. 3—Expanding about Row

- Expanding by the second row, we get:

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & -4 \\ -2 & 5 & 6 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= -0\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 0 + 2\left[2\cdot 6 - (-1)(-2)\right] - 4\left[2.5 - 3(-2)\right]$$

$$= 0 + 20 - 64 = -44$$

E.g. 3—Expanding about Column

- Expanding by the third column, we get:

Example (b)

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= -[0 \cdot 5 - 2(-2)] - 4[2 \cdot 5 - 3(-2)]$$

$$+6[2.2 - 3 \cdot 0]$$

$$= -4 - 64 + 24 = -44$$

E.g. 3—Expanding Determinant about Row and Column

- In both cases, we obtain the same value for the determinant as when we expanded by the first row in Example 2.

Inverse of Square Matrix

- The following criterion allows us to determine whether a square matrix has an inverse without actually calculating the inverse.
  - This is one of the most important uses of the determinant in matrix algebra.
  - It is reason for the name determinant.

Invertibility Criterion

- If A is a square matrix, then A has an inverse if and only if $\det(A) \neq 0$.
  - We will not prove this fact.
  - However, from the formula for the inverse of a 2 x 2 matrix, you can see why it is true in the 2 x 2 case.

E.g. 4—Determinant to Show Matrix Is Not Invertible

- Show that the matrix A has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{bmatrix}$$

  - We begin by calculating the determinant of A.
  - Since all but one of the elements of the second row is zero, we expand the determinant by the second row.

E.g. 4—Determinant to Show Matrix Is Not Invertible

- If we do so, we see from this equation that only the cofactor A_{24} needs to be calculated.

$$\det(A)$$

$$= \begin{vmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= -0 \cdot A_{21} + 0 \cdot A_{22} - 0 \cdot A_{23} + 3 \cdot A_{24}$$

$$= 3A_{24}$$

E.g. 4—Determinant to Show Matrix Is Not Invertible

$$= 3\begin{vmatrix} 1 & 2 & 0 \\ 5 & 6 & 2 \\ 2 & 4 & 0 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= 3(-2)\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= 3(-2)(1 \cdot 4 - 2 \cdot 2)$$

$$= 0$$

- Since the determinant of *A* is zero, *A* cannot have an inverse—by the Invertibility Criterion.

- Row and Column Transformations

Row and Column Transformations

- The preceding example shows that, if we expand a determinant about a row or column that contains many zeros, our work is reduced considerably.
  - We don’t have to evaluate the cofactors of the elements that are zero.

Row and Column Transformations

- The following principle often simplifies the process of finding a determinant by introducing zeros into it without changing its value.

Row and Column Transformations of a Determinant

- If A is a square matrix, and if the matrix B is obtained from A by adding a multiple of one row to another, or a multiple of one column to another, then

$$\det(A) = \det(B)$$

E.g. 5—Using Row and Column Transformations

- Find the determinant of the matrix *A*.

$$A = \begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 24 & 6 & 1 & -12 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

  – Does it have an inverse?

E.g. 5—Using Row and Column Transformations

- If we add –3 times row 1 to row 3, we change all but one element of row 3 to zeros:

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

  – This new matrix has the same determinant as *A*.

E.g. 5—Using Row and Column Transformations

- If we expand its determinant by the third row, we get:

$$\det(A) = 4\begin{vmatrix} 8 & 2 & -4 \\ 3 & 5 & 11 \\ 2 & 2 & -1 \end{vmatrix}$$

  – Now, adding 2 times column 3 to column 1 in this determinant gives us the following result.

E.g. 5—Using Row and Column Transformations

$$\det(A) = 4\begin{vmatrix} 0 & 2 & -4 \\ 25 & 5 & 11 \\ 0 & 2 & -1 \end{vmatrix} = 4(-25)\begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix}$$

$$= 4(-25)\left[2(-1) - (-4)2\right]$$

$$= -600$$

– Since the determinant of *A* is not zero, *A* does have an inverse.