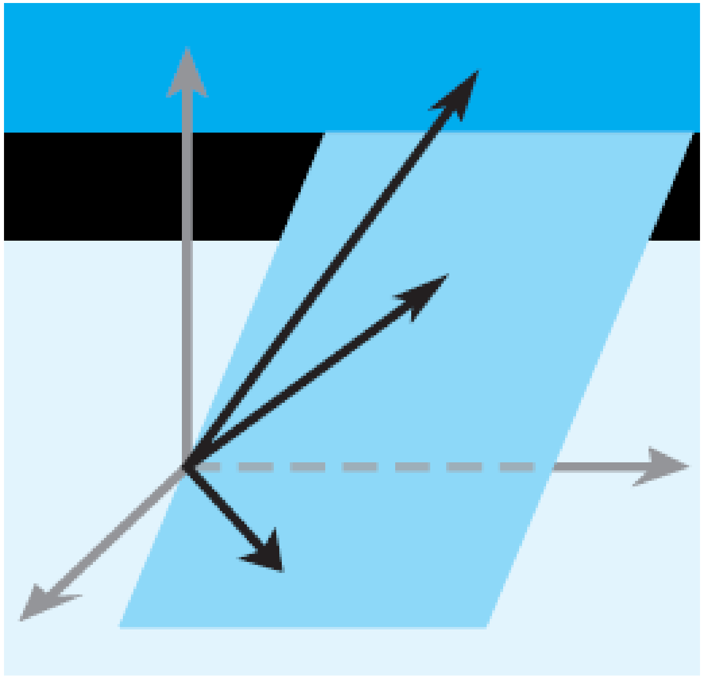
Faculty of Engineering
Mechanical Engineering Department

# Linear Algebra and Vector Analysis
# MATH 1120
# Lecture 13

# Elementary Linear Algebra



Chapter 3

Howard Anton

# Properties of Vectors

**THEOREM 3.1.1** *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* $R^n$, *and if* $k$ *and* $m$ *are scalars, then:*

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

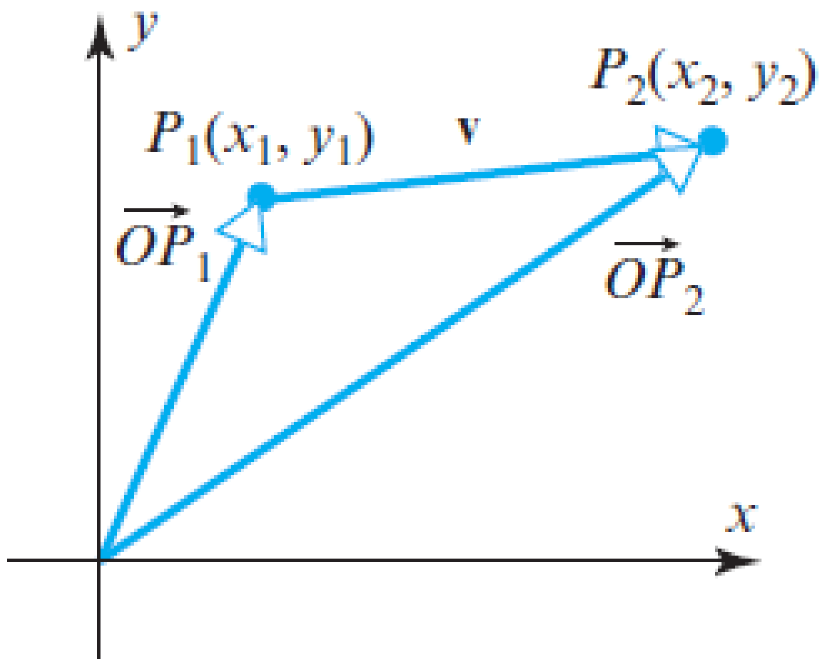
(*e*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*f*) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(*g*) $k(m\mathbf{u}) = (km)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

# *Vectors Whose Initial Point Is Not at the Origin*




$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

- we need to discuss briefly how to generate a vector given the initial and final points of the representation. Given the two points

$$A = (a_1, a_2, a_3) \text{ and } B = (b_1, b_2, b_3)$$

$\overrightarrow{AB}$ is,

$$\vec{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

***Example 1*** Give the vector for each of the following.

**(a)** The vector from $(2,-7,0)$ to $(1,-3,-5)$.

**(b)** The vector from $(1,-3,-5)$ to $(2,-7,0)$.

**(c)** The position vector for $(-90,4)$

***Solution***

**(a)** Remember that to construct this vector we subtract coordinates of the starting point from the ending point.

$$\langle 1-2,-3-(-7),-5-0\rangle = \langle -1,4,-5\rangle$$

**(b)** Same thing here.

$$\langle 2-1,-7-(-3),0-(-5)\rangle = \langle 1,-4,5\rangle$$

find the components of the vector $\overrightarrow{P_1P_2}$.

(a) $P_1(3, 5), \quad P_2(2, 8)$

(b) $P_1(5, -2, 1), \quad P_2(2, 4, 2)$

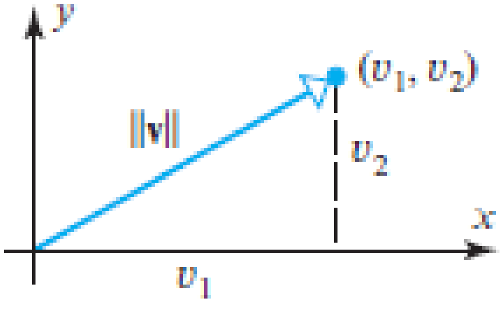
**Answer:**

(a) $\overrightarrow{P_1P_2} = (-1, 3)$

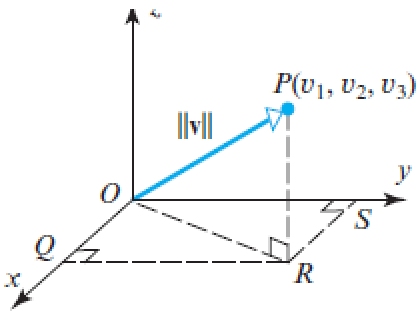
(b) $\overrightarrow{P_1P_2} = (-3, 6, 1)$

# Section 3.2 Norm, Dot Product, and Distance in $R^n$

**The *norm of v, the length of v, or the magnitude of v (the term "norm" being a common* mathematical synonym for length)**



(a)



(b)

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

# Section 3.2 Norm, Dot Product, and Distance in $R^n$

## Norm:

**DEFINITION 1** If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $R^n$, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

## Unit Vectors:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

# Example:

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4)$$

$$-\mathbf{w} = (-4, -2, -1), \qquad \mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)$$

*Example 2* Determine the magnitude of each of the following vectors.

**(a)** $\vec{a} = \langle 3, -5, 10 \rangle$

**(b)** $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$

**(c)** $\vec{w} = \langle 0, 0 \rangle$

**(d)** $\vec{i} = \langle 1, 0, 0 \rangle$

***Solution***

There isn't too much to these other than plug into the formula.

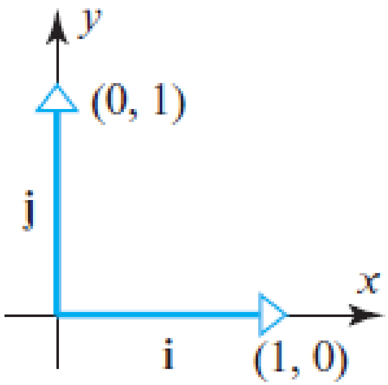
**(a)** $\|\vec{a}\| = \sqrt{9 + 25 + 100} = \sqrt{134}$

**(b)** $\|\vec{u}\| = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1$

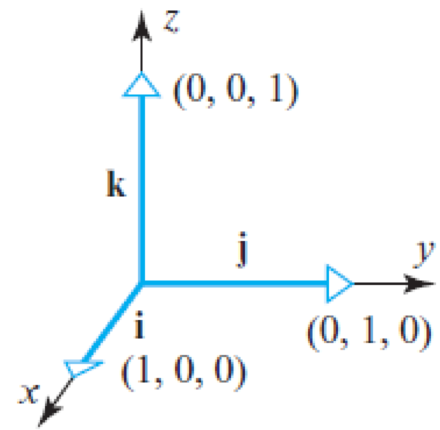
**(c)** $\|\vec{w}\| = \sqrt{0 + 0} = 0$

**(d)** $\|\vec{i}\| = \sqrt{1 + 0 + 0} = 1$

# The Standard Unit Vectors




(a)




(b)

vectors are denoted by

$$\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1)$$

and in $R^3$ by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1)$$

***Example 1*** For the vector $\vec{a} = \langle 2, 4 \rangle$ compute $3\vec{a}$ , $\frac{1}{2}\vec{a}$ and $-2\vec{a}$ . Graph all four vectors on the same axis system.

***Solution***

Here are the three scalar multiplications.

$$3\vec{a} = \langle 6, 12 \rangle \qquad \frac{1}{2}\vec{a} = \langle 1, 2 \rangle \qquad -2\vec{a} = \langle -4, -8 \rangle$$

***Example 2*** Determine if the sets of vectors are parallel or not.

**(a)** $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

**(b)** $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

***Solution***

**(a)** These two vectors are parallel since $\vec{b} = -3\vec{a}$

**(b)** These two vectors aren't parallel. This can be seen by noticing that $4\left(\frac{1}{2}\right) = 2$ and yet $10\left(\frac{1}{2}\right) = 5 \neq -9$. In other words we can't make $\vec{a}$ be a scalar multiple of $\vec{b}$.

***Example 3*** Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2, 1 \rangle$.

Here's what we'll do. First let's determine the magnitude of $\vec{w}$.

$$\|\vec{w}\| = \sqrt{25 + 4 + 1} = \sqrt{30}$$

Now, let's form the following new vector,

$$\vec{u} = \frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{\sqrt{30}}\langle -5, 2, 1 \rangle = \left\langle -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right\rangle$$

The claim is that this is a unit vector. That's easy enough to check

$$\|\vec{u}\| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = \sqrt{\frac{30}{30}} = 1$$