Basic Rules of Differentiation

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Basic Rules of Differentiation

1. Using the definition of the derivative, can be tiresome. In this lesson we are going to learn and use some basic rules of differentiation that are derived from the definition. For these rules, let's assume that we are discussing differentiable functions.

2. These are two ways to denote derivative

$$1.\ f'(x)$$

$$2.\ \frac{df(x)}{dx}$$

Rule 1: Derivative of a constant

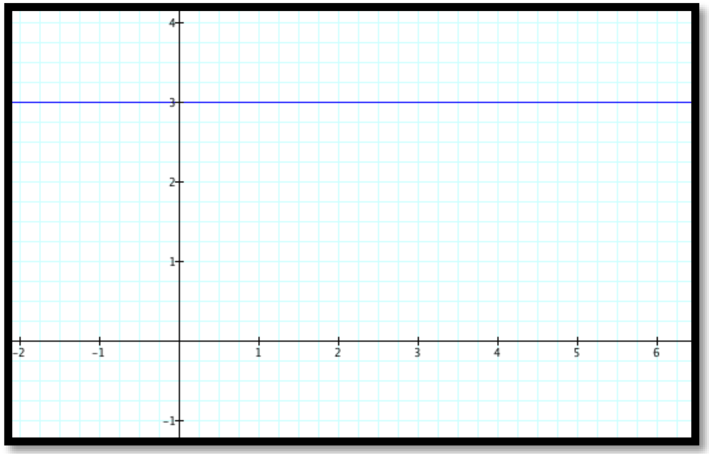
$$\frac{d(c)}{dx} = 0 \qquad \text{for any constant } c$$

This rule states that the derivative of a constant is zero.

For example,

$$f(x) = 5$$

$$f'(x) = 0$$



Note: The constant function is a horizontal line with a constant slope of 0.

Rule 2: The Power Rule

$$\frac{d\left(x^n\right)}{dx} = nx^{n-1}$$, where n is any real number

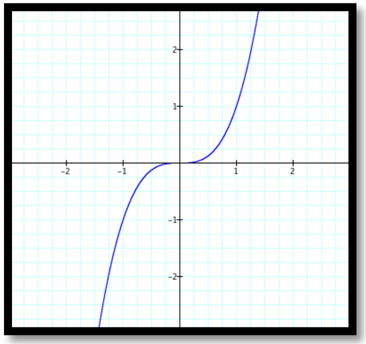
This rule states that the derivative of x raised to a power is the power times x raised to a power one less or n-1.

For example,

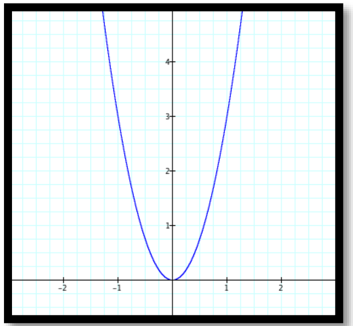
$$f\left(x\right) = x^5$$

$$f'\left(x\right) = 5x^4$$

$$f\left(x\right) = x^3$$



$$f'\left(x\right) = 3x^2$$



Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$, where c is a constant

This rule states that the derivative of a constant times a function is the constant times the derivative of the function.

For example, find the derivative of $f(x) = 5x^4$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(5x^4)$$
$$= 5\frac{d}{dx}(x^4)$$
$$= 5(4x^3)$$
$$= 20x^3$$

Rule 4: Derivative of a Sum or Difference

$$\boxed{\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]}$$

This rule states that the derivative of a sum or difference is the sum or difference of the derivatives.

For example, find the derivative of *f(x)* = x^2 +2*x* -3

$$\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(2x) - \frac{d}{dx}(3)$$
$$= 2x + 2 - 0$$
$$= 2x + 2$$

Continue …

$$f(x)=\left(x^8+12x^5-4x^4+10x^3-6x+5\right)$$

Solution

$$f'(x)=\left(x^8\right)+\tfrac{d}{dx}\left(12x^5\right)-\tfrac{d}{dx}\left(4x^4\right)+\tfrac{d}{dx}\left(10x^3\right)-\tfrac{d}{dx}(6x)+\tfrac{d}{dx}(5)$$

$$=\left(x^8\right)+12\tfrac{d}{dx}\left(x^5\right)-4\tfrac{d}{dx}\left(x^4\right)+10\tfrac{d}{dx}\left(x^3\right)-6\tfrac{d}{dx}(x)+\tfrac{d}{dx}(5)$$

$$=8x^{8-1}+12\cdot 5x^{5-1}-4\cdot 4x^{4-1}+10\cdot 3x^{3-1}-6\cdot 1x^{1-1}+0$$

$$=8x^7+60x^4-16x^3+30x^2-6$$

More Examples: Find $\dfrac{d}{dx}\left(\sqrt{x}\right)$

You'll notice none of the basic rules specifically mention radicals, so you should convert the radical to its exponential form, $x^{1/2}$ and then use the power rule.

Solution

$$f(x) = \sqrt{x}$$

$$\left(\sqrt{x}\right) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{2}} \right)$$

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}}$$

$$f'(x) = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Find derivative

$$f(x) = \left(\frac{1}{x^2} \right)$$

Again, you need to rewrite the expression so that you can use one of the basic rules for differentiation. If we rewrite the fraction as x^{-2} ,then we can use the power rule.

$$f'(x) = \frac{d}{dx} \left(\frac{1}{x^2} \right)$$

$$f'(x) = \frac{d}{dx} \left(x^{-2} \right) = -2x^{-2-1}$$

$$f'(x) = -2x^{-3} = \frac{-2}{x^3}$$

Find derivative

$$f(x)=\left(\frac{4x^3-2x+7}{x}\right)$$

Rewrite the expression so that you can use the basic rules of differentiation.

$$\frac{4x^3-2x+7}{x}=\frac{4x^3}{x}-\frac{2x}{x}+\frac{7}{x}=4x^2-2+7x^{-1}$$

Solution

$$\frac{d}{dx}\left(\frac{4x^3-2x+7}{x}\right)=\frac{d}{dx}\left(4x^2-2+7x^{-1}\right)$$

$$=\frac{d}{dx}\left(4x^2\right)-\frac{d}{dx}(2)+\frac{d}{dx}\left(7x^{-1}\right)$$

$$=4\cdot 2x-0+7\cdot -1x^{-1-1}$$

$$=8x-7x^{-2}$$

$$=8x-\frac{7}{x^2}$$

APPLICATIONS

Q: The position of a particle is given by the equation $s(t) = t^3 - 6t^2 + 9t$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

Solution

$$s'(t) = v(t) = \tfrac{d}{dt}\left(t^3 - 6t^2 + 9t\right)$$

NOTE: The derivative of the position function is the velocity function.

$$v(t) = \tfrac{d}{dt}\left(t^3\right) - \tfrac{d}{dt}\left(6t^2\right) + \tfrac{d}{dt}(9t)$$

$$v(t) = \tfrac{d}{dt}\left(t^3\right) - 6\tfrac{d}{dt}\left(t^2\right) + 9\tfrac{d}{dt}\left(t^1\right)$$

$$v(t) = 3t^{3-1} - 6\cdot 2t^{2-1} + 9\cdot t^{1-1}$$

$$v(t) = 3t^2 - 12t + 9$$

Continue …

(b) What is the velocity after 2 seconds?

$$v(2) = 3(2)^2 - 12(2) + 9$$
$$v(2) = -3m/s$$

(c) What is the speed after 2 seconds?

$$v(2) = -3m/s$$
$$|v(2)| = 3m/s$$

(d) When is the particle at rest?

$$v(t) = 3t^2 - 12t + 9$$
$$0 = 3t^2 - 12t + 9$$
$$0 = 3(t-1)(t-3)$$
$$t = 1, 3$$

Note: The particle is at rest when the velocity is 0.

After 1 second and 3 seconds velocity will be zero

Continue …

Q; Find the slope and equation of the following tangent line to the curve at the point (1,3).

$$y = 2x^2 + 1$$

- Step 1: The derivative gives the slope of the tangent to the curve. So we will need to find the derivative and evaluate it at x = 1 to find the slope at the point (1,3).
- Step 2: Then we'll use the slope and the point to write the equation of the tangent line using the point slope form.

Step 1

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^2 + 1\right) = \frac{d}{dx}\left(2x^2\right) + \frac{d}{dx}\left(1\right)$$

$$= 2 \cdot 2x^{2-1} + 0$$

$$= 4x$$

Now, evaluate the derivative at $x = 1$ to find the slope (m) at (1,3).

$$m = 4 \cdot 1 = 4$$

Continue …

Step 2: Start with the point slope form and use the slope, 4 and the point (1,3).

$$m = \frac{y - y_1}{x - x_1}$$

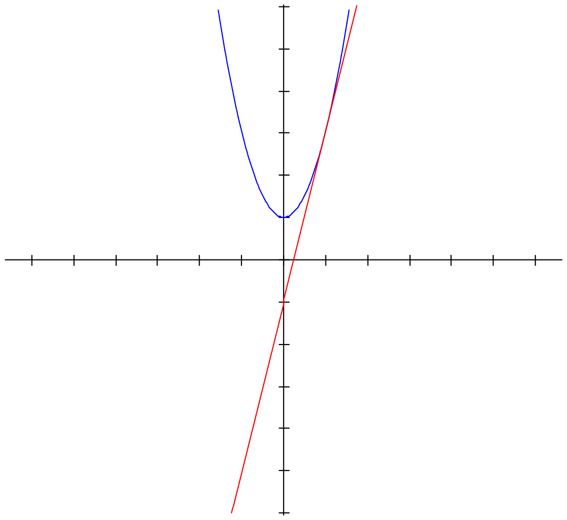
$$y - y_1 = m(x - x_1)$$

$$y - 3 = 4(x - 1)$$

$$y - 3 = 4x - 4$$

$$y = 4x - 1$$

The graph below shows the curve $y = 2x^2 + 1$ in blue and the tangent line at the point (1,3), $y = 4x - 1$ in red.



Q1: Find the slope (m) of $f(x)$ at $x = 3$, $x = -2$

Solutions

$$f(x) = 3x^2 - 4x + 7$$
$$f'(x) = 6x - 4$$
$$f'(3) = 6(3) - 4 = 14$$
$$f'(-2) = 6(-2) - 4 = -16$$

Q2: Write the equation of the tangent line at $x = -2$ for above function

Solutions

$$m = \frac{y - y_1}{x - x_1}$$
$$y - y_1 = m(x - x_1)$$

$$f(-2) = 3(-2)^2 - 4(-2) + 7 = 27$$
$$y - 27 = -16(x + 2)$$
$$y = 27 + -16(x + 2)$$
$$\boxed{y = -16x - 5}$$

Continue …

Q: Find all the *x* values where $y = x^3 + 2x^2 + x$ has a horizontal tangent line.

Solutions

Step 1: Find the derivative.

$$\frac{d}{dx}\left(x^3 + 2x^2 + x\right) = 3x^2 + 4x + 1$$

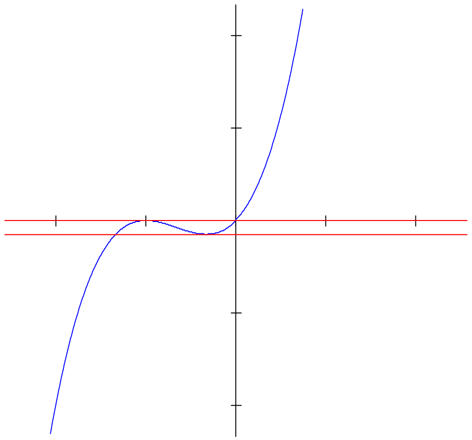
Step 2: Since horizontal lines have a slope of 0, set the derivative equal to 0 and solve for *x*.

$$3x^2 + 4x + 1 = 0$$

$$(3x+1)(x+1) = 0$$

$$3x + 1 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = \frac{-1}{3} \quad \text{or} \quad x = -1$$



Therefore, the *x* values where the function has horizontal tangents is at *x* = -1 , -1/3.

Continue …

Q: Find the horizontal tangents of: $y = x^4 - 2x^2 + 2$

Solution

$$y = x^4 - 2x^2 + 2$$

$$\frac{dy}{dx} = 4x^3 - 4x$$

Horizontal tangents occur when slope = zero.

$$4x^3 - 4x = 0$$

$$x^3 - x = 0$$

$$x\left(x^2 - 1\right) = 0$$

$$x(x+1)(x-1) = 0$$

$$x = 0, -1, \ 1$$

Plugging the x values into the original equation, we get:

$$y = 2, \ y = 1, \ y = 1$$

We got two horizontal tangents.

$$(-1, 1), \ (1, 1)$$

Differentiation Rules: Product Rule

$$\frac{d}{dx}\left(uv\right) = u\frac{dv}{dx} \quad v\frac{du}{dx}$$

Example

$$f(x) = \left(x^3 + 2x + 5\right)\left(3x^7 - 8x^2 + 1\right)$$

$$f'(x) = \left(3x^2 + 2\right)\left(3x^7 - 8x^2 + 1\right) + \left(x^3 + 2x + 5\right)\left(21x^6 - 16x\right)$$

Derivative of the first function

Derivative of the second function

$$f'(x) = 30x^9 + 48x^7 + 105x^6 - 40x^4 - 45x^2 - 80x + 2$$

Q: Find derivative of the following function

Solution

$$f(x)=\left(x^2+3\right)\left(2x^3+5x\right)$$

$$f'(x)=\left(x^2+3\right)\frac{d}{dx}\left[2x^3+5x\right]+(2x^3+5x)\frac{d}{dx}\left[x^2+3\right]$$

$$f'(x)=\left(x^2+3\right)\left(6x^2+5\right)+(2x^3+5x)\left(2x+0\right)$$

$$=6x^4+5x^2+18x^2+15+4x^4+10x^2$$

$$f'(x)=10x^4+33x^2+15$$

Differentiation Rules: Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$$

Example: Find derivative of the following function

$$f(x)=\left(\frac{2x^3+5x}{x^2+3}\right)$$

Solution

$$\frac{d}{dx}\left(\frac{2x^3+5x}{x^2+3}\right)=\frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

Q: Find derivative of the following function

$$f(x) = \frac{3x+5}{x^2-2}$$

Solution

Derivative of the numerator

Derivative of the denominator

$$f'(x) = \frac{3\left(x^2-2\right) - 2x\left(3x+5\right)}{\left(x^2-2\right)^2}$$

$$= \frac{-3x^2 - 10x - 6}{\left(x^2-2\right)^2}$$

Q: Find derivative of the following functions

$$f(x)=(3x-2x^2)(5+4x)$$

Solution

$$f'(x)=(3x-2x^2)\frac{d}{dx}(5+4x)+(5+4x)\frac{d}{dx}(3x-2x^2)$$

$$=(3x-2x^2)4+(5+4x)(3-4x)$$

$$=-24x^2+4x+15$$

$$f(x)=\frac{5x-2}{x^2+1}$$

Solution

$$f'(x)=\frac{(x^2+1)\frac{d}{dx}(5x-2)-(5x-2)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=\frac{(x^2+1)5-(5x-2)2x}{(x^2+1)^2}=\frac{(5x^2+5)-(10x^2-4x)}{(x^2+1)^2}=\frac{-5x^2+4x+5}{(x^2+1)^2}$$

Differentiation Rules: Chain Rule

If $h(x) = [f(x)]^n$ (n, real) then

$$\boxed{h'(x) = n[f(x)]^{n-1} \cdot f'(x)}$$

Q: Find derivative of the following function

$$f(x) = \sqrt{3x^2 + 4x} = \left(3x^2 + 4x\right)^{1/2}$$

$$f'(x) = \frac{1}{2}\left(3x^2 + 4x\right)^{-1/2}(6x + 4)$$

$$f'(x) = \frac{3x + 2}{\sqrt{3x^2 + 4x}}$$

Q: Find derivative of the following function

$$G(x) = \left(\frac{2x-1}{3x+5}\right)^7$$

Solution

$$G'(x) = 7\left(\frac{2x-1}{3x+5}\right)^6 \left(\frac{(3x+5)2-(2x-1)3}{(3x+5)^2}\right)$$

$$G'(x) = 7\left(\frac{2x-1}{3x+5}\right)^6 \frac{13}{(3x+5)^2} = \frac{91(2x-1)^6}{(3x+5)^8}$$

Q: Find derivative of the following function

$$y = u^{5/2},\ u = 7x^8 + 3x^2$$

Solution

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{5}{2} u^{3/2} \cdot \left(56x^7 + 6x\right)$$

Sub in for u

$$= \frac{5}{2}\left(7x^8 + 3x^2\right)^{3/2} \cdot \left(56x^7 + 6x\right)$$

$$= \left(140x^7 + 15x\right)\left(7x^8 + 3x^2\right)^{3/2}$$

Derivatives of Trigonometric functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

$$\frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

cosecant *θ* (csc)is the reciprocal of **sine *θ***
secant *θ* (sec) is the reciprocal of **cosine *θ***
cotangent *θ* (cot) is the reciprocal of **tangent *θ***

Q: Find derivative of tan(x) by using the quotient rule.

Solution

$$\tan x = \frac{\sin x}{\cos x}$$

$$f(x) = \tan x$$

$$f'(x) = \frac{d(\frac{\sin x}{\cos x})}{dx}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$f'(x) = \sec^2 x$$

Find the derivatives of (a) $y = x^2 \sin x$ (b) $y = \ln(x^2 \sin x)$ (c) $y = \dfrac{\cos x}{1-\sin x}$

Solution: Using product rule which we have already studied

(a) $y = x^2 \sin x$

$$\frac{dy}{dx} = x^2 \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx} x^2$$

$$= x^2 \times \cos x + \sin x \times (2x)$$

$$= x^2 \cos x + 2x \sin x$$

(b) $y = \ln(x^2 \sin x)$

$$y' = \frac{d}{dx}\left(\ln(x^2 \sin x)\right) \times \frac{d}{dx}\left(x^2 \sin x\right)$$

$$= \frac{1}{x^2 \sin x} \times \left(x^2 \cos x + 2x \sin x\right)$$

$$= \frac{x^2 \cos x}{x^2 \sin x} + \frac{2x \sin x}{x^2 \sin x}$$

$$= \frac{\cos x}{\sin x} + \frac{2}{x}$$

$$= \frac{2}{x} + \tan x$$

Solution (c)

$$y=\frac{\cos x}{1-\sin x}$$

Using quotient rule which we have already studied

$$\frac{dy}{dx}=\frac{(1-\sin x)\times\frac{d}{dx}\cos x-\cos x\times\frac{d}{dx}(1-\sin x)}{(1-\sin x)^{2}}$$

$$=\frac{(1-\sin x)(-\sin x)-\cos x(0-\cos x)}{(1-\sin x)^{2}}$$

$$=\frac{-\sin x+\sin^{2}x+\cos^{2}x}{(1-\sin x)^{2}}$$

$$=\frac{1-\sin x}{(1-\sin x)^{2}}$$

$$=\frac{1-\sin x}{(1-\sin x)}$$

Q: Find the derivatives of the following functions

A

$$f(x) = 5\sin x - \frac{1}{2}\sec x + x\tan x - 7x^2 + 3$$

$$f'(x) = 5\cos x - \frac{1}{2}\sec x\tan x + x\sec^2 x + \tan x(1) - 14x$$

B

$$f(x) = \frac{1+\sin x}{x+\cos x}$$

$$f'(x) = \frac{(x+\cos x)\frac{d}{dx}(1+\sin x) - (1+\sin x)\frac{d}{dx}(x+\cos x)}{(x+\cos x)^2}$$

$$f'(x) = \frac{(x+\cos x)(\cos x) - (1+\sin x)(1-\sin x)}{(x+\cos x)^2}$$

$$f'(x) = \frac{(x\cos x + \cos^2 x) - (1-\sin^2 x)}{(x+\cos x)^2} = \frac{x\cos x + \cos^2 x - 1 + \sin^2 x}{(x+\cos x)^2}$$

$$f'(x) = \frac{x\cos x}{(x+\cos x)^2}$$

Higher Order Derivatives

$$y' = \frac{dy}{dx}$$ is the first derivative of y with respect to x.

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$ is the second derivative.

(y double prime)

$$y''' = \frac{dy''}{dx}$$ is the third derivative.

$$y'''' = \frac{d}{dx}y'''$$ is the fourth derivative.

Example of Higher Derivatives

Given $f(x) = 3x^5 - 2x^3 + 14$ find $f'''(x)$.

$$f'(x) = 15x^4 - 6x^2$$

$$f''(x) = 60x^3 - 12x$$

$$\boxed{f'''(x) = 180x^2 - 12}$$

Continue …

Given $f(x)=\dfrac{2x+1}{3x-2}$ find $f''(2)$.

$$f'(x)=\frac{2\left(3x-2\right)-3\left(2x+1\right)}{\left(3x-2\right)^2}=\frac{-7}{\left(3x-2\right)^2}=-7\left(3x-2\right)^{-2}$$

$$f''(x)=14\left(3x-2\right)^{-3}\left(3\right)=\frac{42}{\left(3x-2\right)^3}$$

$$\boxed{f''(2)=\frac{42}{\left(3(2)-2\right)^3}=\frac{42}{4}=\frac{21}{32}}$$

Q: Find y'' if $y = \sec x$.

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

$$= \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$