جامعة الملك سعود
King Saud University

Linear Algebra and Vector Analysis
MATH 1120
Applied Engineering, Tutorial (2):
SOLUTION

1- Find a linear system in the unknowns $x1$, $x2$, $x3$, . . . , that corresponds to the given augmented matrix.

a-

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(a)
$$\begin{aligned} 2x_1 \qquad &= 0 \\ 3x_1 - 4x_2 &= 0 \\ x_2 &= 1 \end{aligned}$$

b-

(b) $$\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$$

(b)
$$\begin{aligned} 3x_1 \qquad\quad - 2x_3 &= 5 \\ 7x_1 + x_2 + 4x_3 &= -3 \\ -2x_2 + x_3 &= 7 \end{aligned}$$

2- Find the augmented matrix for the linear system.

(a)
$$\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} 2x_2 \qquad - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 \qquad\qquad &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

(a) $\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$$

3-

For the following augmented matrix perform the indicated elementary row operations.

$$\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

(a) $8R_1$ (b) $R_2 \leftrightarrow R_3$ (c) $R_2 + 3R_1 \to R_2$

(a) $8R_1$

This operation is telling us to multiply all the entries in Row 1 of the augmented matrix by 8 so let's do that.

$$\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \overset{8R_1}{\to} \left[\begin{array}{ccc|c} 32 & -8 & 24 & 40 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

(b) $R_2 \leftrightarrow R_3$

This operation is telling us to interchange Row 2 and Row 3 of the augmented matrix. Here is that work.

$$\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\to} \left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ -6 & 1 & -3 & 10 \\ 0 & 2 & 5 & 9 \end{array}\right]$$

(c) $R_2 + 3R_1 \to R_2$

For this operation we are going to replace Row 2 with the results of taking the original entries from Row 2 and add to them 3 times the entries in Row 1.

$$\left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 0 & 2 & 5 & 9 \\ -6 & 1 & -3 & 10 \end{array}\right] \overset{R_2 + 3R_1 \to R_2}{\to} \left[\begin{array}{ccc|c} 4 & -1 & 3 & 5 \\ 12 & -1 & 14 & 24 \\ -6 & 1 & -3 & 10 \end{array}\right]$$

4-

For the following augmented matrix perform the indicated elementary row operations.

$$\left[\begin{array}{ccc:c} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

(a) $\frac{1}{2}R_2$ (b) $R_1 \leftrightarrow R_3$ (c) $R_1 - 6R_3 \to R_1$

(a) $\frac{1}{2}R_2$

This operation is telling us to multiply all the entries in Row 2 of the augmented matrix by ½ so let's do that.

$$\left[\begin{array}{ccc:c} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc:c} 1 & -6 & 2 & 0 \\ 1 & -4 & 5 & 2 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

(b) $R_1 \leftrightarrow R_3$

This operation is telling us to interchange Row 1 and Row 3 of the augmented matrix. Here is that work.

$$\left[\begin{array}{ccc:c} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc:c} 3 & -4 & -1 & 2 \\ 2 & -8 & 10 & 4 \\ 1 & -6 & 2 & 0 \end{array}\right]$$

(c) $R_1 - 6R_3 \to R_1$

For this operation we are going to replace Row 1 with the results of taking the original entries from Row 1 and subtract from them 6 times the entries in Row 3.

$$\left[\begin{array}{ccc:c} 1 & -6 & 2 & 0 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right] \xrightarrow{R_1 - 6R_3 \to R_1} \left[\begin{array}{ccc:c} -17 & 18 & 8 & -12 \\ 2 & -8 & 10 & 4 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

5- For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent

$$x-7y=-11$$
$$5x+2y=-18$$

Step 1
The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 5 & 2 & -18 \end{array}\right]$$

Step 2
We need to make the number in the upper left corner a one. In this case it already is and so there really isn't anything to do in this step for this particular problem.

Step 3
Next, we need to convert the 5 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 5 & 2 & -18 \end{array}\right] \quad \underset{\rightarrow}{R_2-5R_1 \to R_2} \quad \left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 37 & 37 \end{array}\right]$$

Step 4
The next step is to turn the number at the bottom of the second column (37 in this case) into a one. The following elementary row operation will do that for us.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 37 & 37 \end{array}\right] \quad \underset{\rightarrow}{\tfrac{1}{37}R_2} \quad \left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 1 & 1 \end{array}\right]$$

Step 5
Finally we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & -7 & -11 \\ 0 & 1 & 1 \end{array}\right] \quad \underset{\rightarrow}{R_1+7R_2 \to R_1} \quad \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right]$$

Step 6
From the final augmented matrix we found in Step 5 we get the solution to the system is : $\boxed{x=-4,\ y=1}$

6- For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$7x-8y=-12$$
$$-4x+2y=3$$

Step 1
The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{cc|c} 7 & -8 & -12 \\ -4 & 2 & 3 \end{array}\right]$$

Step 2
We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation $\frac{1}{7}R_1$. However, this would put fractions into the other two entries in the first row and it might be nice to avoid them.

So, instead let's do the following elementary row operation.

$$\left[\begin{array}{cc|c} 7 & -8 & -12 \\ -4 & 2 & 3 \end{array}\right] \xrightarrow{R_1+2R_2 \to R_1} \left[\begin{array}{cc|c} -1 & -4 & -6 \\ -4 & 2 & 3 \end{array}\right]$$

Now, this isn't quite what we want since the number in the upper left is a minus one and not a positive one. However, we can easily fix that by multiplying the first row by -1.

$$\left[\begin{array}{cc|c} -1 & -4 & -6 \\ -4 & 2 & 3 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ -4 & 2 & 3 \end{array}\right]$$

Note that as this step has shown there are several different paths to do these problems. Some will result in "messier" intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3
Next, we need to convert the -4 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ -4 & 2 & 3 \end{array}\right] \xrightarrow{R_2+4R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 18 & 27 \end{array}\right]$$

Step 4
The next step is to turn the number at the bottom of the second column (18 in this case) into a one. The following elementary row operation will do that for us.

$$\left[\begin{array}{cc|c}1 & 4 & 6\\ 0 & 18 & 27\end{array}\right] \quad \underset{\rightarrow}{\tfrac{1}{18}R_2} \quad \left[\begin{array}{cc|c}1 & 4 & 6\\ 0 & 1 & \tfrac{27}{18}\end{array}\right] = \left[\begin{array}{cc|c}1 & 4 & 6\\ 0 & 1 & \tfrac{3}{2}\end{array}\right]$$

In the first step we chose to avoid the step that put fractions into the augmented matrix, but sometimes, as in this step, they can't be avoided.

Step 5
Finally we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

$$\left[\begin{array}{cc|c}1 & 4 & 6\\ 0 & 1 & \tfrac{3}{2}\end{array}\right] \quad \underset{\rightarrow}{R_1 - 4R_2 \rightarrow R_1} \quad \left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & \tfrac{3}{2}\end{array}\right]$$

Step 6
From the final augmented matrix we found in Step 5 we get the solution to the system is : $\boxed{x = 0,\ y = \tfrac{3}{2}}$.

7- For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$3x + 9y = -6$$
$$-4x - 12y = 8$$

Step 1
The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{cc|c}3 & 9 & -6\\ -4 & -12 & 8\end{array}\right]$$

Step 2
We need to make the number in the upper left corner a one. In this case we can quickly do that by dividing the top row by 3.

$$\left[\begin{array}{cc|c}3 & 9 & -6\\ -4 & -12 & 8\end{array}\right] \quad \underset{\rightarrow}{\tfrac{1}{3}R_1} \quad \left[\begin{array}{cc|c}1 & 3 & -2\\ -4 & -12 & 8\end{array}\right]$$

Step 3
Next, we need to convert the -4 below the 1 into a zero and we can do that with the following elementary row operation.

$$\left[\begin{array}{cc|c}1 & 3 & -2\\ -4 & -12 & 8\end{array}\right] \quad \underset{\rightarrow}{R_2 + 4R_1 \rightarrow R_2} \quad \left[\begin{array}{cc|c}1 & 3 & -2\\ 0 & 0 & 0\end{array}\right]$$

Step 4
The minute we see the bottom row of all zeroes we know that the system if **dependent**. We can convert the top row into an equation and solve for x as follows,

$$x+3y=-2 \qquad \rightarrow \qquad x=-3y-2$$

From this we can write the solution as,

$$\begin{aligned} x &= -3t-2 \\ y &= t \end{aligned} \qquad t \text{ is any number}$$

8- For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

$$\begin{aligned} 2x+5y+2z &= -38 \\ 3x-2y+4z &= 17 \\ -6x+y-7z &= -12 \end{aligned}$$

Step 1
The first step is to write down the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c} 2 & 5 & 2 & -38 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right]$$

Step 2
We need to make the number in the upper left corner a one. Much like with the previous problems (*i.e.* solving systems with two variables) we can quickly do it with the elementary row operation $\frac{1}{2}R_1$ but that will put fractions into the augmented matrix and they would probably be around for quite a few steps and it would be really nice to avoid them for as long as possible when the augmented matrix starts getting this size.

So, let's start with the following elementary row operation.

$$\left[\begin{array}{ccc|c} 2 & 5 & 2 & -38 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right] \underset{\rightarrow}{R_1 - R_2 \rightarrow R_1} \left[\begin{array}{ccc|c} -1 & 7 & -2 & -55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right]$$

With this operation we got a negative one in the spot where we needed a plus one, but we can easily fix that with the next elementary row operation.

$$\left[\begin{array}{ccc|c} -1 & 7 & -2 & -55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right] \underset{\rightarrow}{-R_1} \left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right]$$

Now, a quick note before we really jump into the rest of this problem. Using augmented matrices to solve systems with three variables can be a very tedious process and there are a great number of possible paths to take in the solution process so your solution may well vary from this solution depending on the path you took. The final answers however will the same regardless of the path we take provided we did all the arithmetic correctly.

Step 3
Next, we need to convert the 3 and the -6 below the 1 in the first column into zeroes and we can do that with the following elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 3 & -2 & 4 & 17 \\ -6 & 1 & -7 & -12 \end{array}\right] \begin{array}{c} R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 6R_1 \rightarrow R_3 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 19 & -2 & -148 \\ 0 & -41 & 5 & 318 \end{array}\right]$$

Step 4
We now need to turn the 19 in the second row into a one and it seems like the only easy way to do that is the following elementary row operation.

$$\left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 19 & -2 & -148 \\ 0 & -41 & 5 & 318 \end{array}\right] \underset{\rightarrow}{\tfrac{1}{19}R_2} \left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & -41 & 5 & 318 \end{array}\right]$$

Step 5
Next we need to turn the -41 in the third row into a zero. The following elementary row operation will do that for us.

$$\left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & -41 & 5 & 318 \end{array}\right] \underset{\rightarrow}{R_3+41R_2 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & \frac{13}{19} & -\frac{26}{19} \end{array}\right]$$

Again, we had to put more fraction into the augmented matrix. This is just a fact of life with these types of problems. However, as we'll see in the next step they do often disappear as well.

Step 6
Okay, we need to turn the $\frac{13}{19}$ in the third row into a one and we can do that as follows,

$$\left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & \frac{13}{19} & -\frac{26}{19} \end{array}\right] \underset{\rightarrow}{\tfrac{19}{13}R_3} \left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 7
Next we need to turn the $-\frac{2}{19}$ and the 2 in the third column into zeroes. The following elementary row operations will do that for us.

$$\left[\begin{array}{ccc|c} 1 & -7 & 2 & 55 \\ 0 & 1 & -\frac{2}{19} & -\frac{148}{19} \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1-2R_3 \rightarrow R_1 \\ R_2+\frac{2}{19}R_3 \rightarrow R_2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -7 & 0 & 59 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Note that the fractions are now completely gone! This won't always happen but it also will happen fairly regularly that fractions get introduced in intermediate steps and then go away in later steps.

Step 8
For the final operation we need to turn the -7 in the second column into a zero and we can do that as follows,

$$\left[\begin{array}{ccc|c} 1 & -7 & 0 & 59 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right] \underset{\rightarrow}{R_1+7R_2 \rightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 9
From the final augmented matrix we found in Step 8 we get the solution to the system is :
$\boxed{x=3,\ y=-8,\ z=-2}$.