Linear Algebra and Vector Analysis
MATH 1120
Applied Engineering, Tutorial (2): SOLUTION

1- Use Gaussian Elimination and Gauss-Jordan Elimination to solve the following system of linear equations.

$$-2x_1 + x_2 - x_3 = 4$$
$$x_1 + 2x_2 + 3x_3 = 13$$
$$3x_1 + x_3 = -1$$

So, let's start off by getting the augmented matrix for this system.

$$\begin{bmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 - 3R_1 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix} \begin{matrix} \tfrac{1}{5}R_2 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{bmatrix} \xrightarrow{R_3 + 6R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

Summarizing up the solution to the system is,

$x_1 = -1 \qquad x_2 = 4 \qquad x_3 = 2$

This substitution process is called **back substitution**.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - 3R_3} \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -1 \\ x_2 = 4 \\ x_3 = 2 \end{array}$$

2- Solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= 1 \end{aligned}$$

So, here is the augmented matrix for this system.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 + R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ x_2 - x_3 &= -1 \\ 0 &= 1 \end{aligned}$$

There is no solution to this system

3- Solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & -7 \end{bmatrix}$$

From previous problem we can find that

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_3 = -4$$
$$x_2 - x_3 = -1$$

$$x_1 = -4 - x_3$$
$$x_2 = -1 + x_3$$

$x_1 = -4 - t$ $\quad x_2 = -1 + t$ $\quad x_3 = t$ $\quad t$ is any number