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Math 280
"Introduction to real analysis"

Contents :

I - Real numbers (completeness axiom)

II - Sequences (Convergence, Cauchy criterion, subsequences

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IV - Limits of functions

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VI - Differentiation. (Extrema, mean value theorem)

VII - Riemann integral.

VIII - Sequences and series of functions.

References :

1) Elements of real analysis.
M. A. AL-Gwaiz & S. A. Elsanousi
Chapman & Hall / CRC, 2006.

2) Introduction to real analysis. 3rd edition
R. G. Bartle & D. R. Sherbert, Wiley & Sons
2000.

chapter I: "Real numbers"

Field axioms 1-1:

$\mathbb{R}$ denotes the set of all "real" numbers which can be characterized by the "real" line:

on $\mathbb{R}$, we define the two generic operations namely: the addition "$+$" and the multiplication "$\cdot$".

$\mathbb{R}$ endowed with these operations satisfies the following axioms (postulates):

A1: Commutative property of addition:

$$a + b = b + a \, , \ \forall \, a, b \in \mathbb{R}.$$

A2: associative property of addition:

$$a + (b + c) = (a + b) + c \, , \ \forall \, a, b, c \in \mathbb{R}.$$

A3: neutral element of addition:

There is an element $0 \in \mathbb{R}$ such that

$$a + 0 = 0 + a = a \, , \ \forall \, a \in \mathbb{R}.$$

A4: additive inverse:

$\forall \, a \in \mathbb{R}$, there is an element $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.

A5: Commutative property of multiplication:

$a \cdot b = b \cdot a \ , \ \forall a, b \in \mathbb{R}.$

A6: associative property of multiplication:

$a \cdot (b \cdot c) = (a \cdot b) \cdot c \ , \ \forall a, b, c \in \mathbb{R}.$

A7: neutral element for multiplication:

there is an element $1 \neq 0$ in $\mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a \ , \ \forall a \in \mathbb{R}.$

A8: multiplicative inverse:

for every element $a \neq 0$ in $\mathbb{R}$ there is an element a^{-1} in $\mathbb{R}$ such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

A9: Distributive property of multiplication over addition

$a \cdot (b + c) = a \cdot b + a \cdot c \ , \ \forall a, b, c \in \mathbb{R}.$

A10: Assume that there is a subset P of $\mathbb{R}$ such that: for any $a \in \mathbb{R}$ one and only one of the following alternatives holds

$a \in P$, or $a = 0$, or $-a \in P$.

A11: for this P from A10 one has:

if $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$.

Remark 2-1:

1) Axioms A_1, A_2, A_3 and A_4 state that $(\mathbb{R}, +)$

is a group (additive group).

2) Axioms A_5, A_6, A_7 and A_8 state that $(R\setminus\{0\}, \cdot)$ is a group (multiplicative group).

3) Axioms A_{10} and A_{11} are called axioms of order. P is the set of positive numbers R^+, and a relation of order, denoted $>$, is defined as follows:- $a > b$ iff $a - b \in P$

completeness axiom:-

Definition 3-1:
Let A be a subset of R.
1) if there is a real number b such that
$$x \leq b, \ \forall x \in A$$
then b is called an upper bound of A, and A is said to be bounded above.

2) if there is a real number a such that
$$x \geq a, \ \forall x \in A$$
then a is called a lower bound of A, and A is said to be bounded below.

iii) A is said to be bounded, if it is bounded above and below, and unbounded if it is not bounded (either below <u>or</u> above).

Definition 4-1:-
Let A be a subset of $\mathbb{R}$. An element $b \in \mathbb{R}$ is called a least upper bound (l.u.b.), or supremum, of A if
i) b is an upper bound of A, that is $b \geq x$ for all $x \in A$, and
ii) there is no other upper bound of A which is less than b, that is
$$u \geq x, \forall x \in A \Rightarrow u \geq b.$$

Similarly, we call $a \in \mathbb{R}$ a greatest lower bound, or infimum, if
(g.l.b.)
i) a is a lower bound of A, and
ii) there is no lower bound of A which is greater than a, that is
$$u \leq x, \forall x \in A \Rightarrow u \leq a.$$

Example 5-1:-
If A is any of the intervals (a, b), $[a, b)$, $(a, b]$ or $[a, b]$, then
$$\text{Sup } A = a \text{ and } \inf A = a.$$
proof:- Exercise.

Axioms 6.1:- (Completeness axioms)
A12:- If A is a nonempty subset of $\mathbb{R}$ which is bounded above, then it has a least upper bound in $\mathbb{R}$

A12':- If $A \neq \phi$, $A \subset \mathbb{R}$ and bounded below, then it has a greatest lower bound in $\mathbb{R}$.
Lemma 7-1.
There is no rational number x (i.e. $x \in \mathbb{Q}$)

such that $x^2 = 2$.
(Equation $x^2 = 2$ has no solution in $\mathbb{Q}$).

proof :- Exercise.

Theorem 8-1 :-
If $n \in \mathbb{N}$ and $a > 0$, then there is a number $x \in \mathbb{R}$ such that $x^n = a$.
proof :- See the text book P. 4~~1~~ (Theorem 2.6).

Remark 9-1 :-
In case $a = n = 2$, Theorem 8-1 tells us that $\sqrt{2} \in \mathbb{R}$ is a solution of the equation $x^2 = 2$, which has no solution in $\mathbb{Q}$ by lemma ~~7~~-1. Such numbers (like $\sqrt{2}$) are called irrational numbers. More examples about them are :-
$\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, π, e, ...
In general if p is a prime number, then $\sqrt{p}$ is an irrational number.

Theorem 10-1 :- (Archimedes theorem)
The set of natural numbers $\mathbb{N}$ is not bounded.
proof :- See the textbook p. 43.

Corollary 11-1 :-
$\forall x > 0$, $\exists n \in \mathbb{N}$ such that $x > \frac{1}{n}$.
proof :- Exercise.

Corollary 12-1 :-
For every $x \geq 0$, there is an $n \in \mathbb{N}$ such that

$$n - 1 \leq x < n.$$

proof: Exercise.

Theorem 13-1: (Density of $\mathbb{Q}$ in $\mathbb{R}$)
If x and y are real numbers and $x < y$, then there is a rational number r such that $x < r < y$.

Corollary 14-1: (Density of irrationals in $\mathbb{R}$)
$\forall x, y \in \mathbb{R}$ with $x < y$, there is an irrational number l such that
$$x < l < y.$$

Exercise 15-1:
Determine $\sup A$ and $\inf A$ where they exist:
1) $A = \{x \in \mathbb{R},\ x^2 - 9 > 0\}$.
2) $A = \{n \in \mathbb{N},\ 1 - \frac{(-1)^n}{n}\}$.
3) $A = \mathbb{Q}$.
4) $A = \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
5) $A = \mathbb{Z}$.

Exercise 16-1:
If b is an upper bound of A, show that $b = \sup A$ if and only if for every $\varepsilon > 0$ there is an element $a \in A$ such that $a > b - \varepsilon$.
State and prove an analog statement for $\inf A$.

Exercise 17_1 :

Let $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$. Show that $x+y \notin \mathbb{Q}$.
When is $xy \in \mathbb{Q}$.

Exercise 18_1 :

If $x > 0$, show that for every $y \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $nx > y$.

Chapter II : "Sequences"

Sequences and convergence :

Definition 1-2 :

A sequence is a function whose domain is $\mathbb{N}$.
A real sequence is a sequence whose range is $\mathbb{R}$
i.e.

$$x : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longrightarrow x(n) = x_n$$

it is denoted $(x_1, x_2, \dots)$ or $(x_n)_{n \in \mathbb{N}}$.
For example :
the sequence $(2n)$ is just $(2, 4, 6, 8, \dots)$
the sequence $(-1)^n$ is $(-1, 1, -1, 1, \dots)$
the sequence $(\frac{1}{n})$ is $(1, \frac{1}{2}, \frac{1}{3}, \dots)$

Definition 2-2 :

A sequence (x_n) is said to be convergent if there exits a real number x with the following property : for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies $|x_n - x| < \varepsilon$.
If (x_n) is not convergent it is called divergent.
We write

$$\lim_{n \to \infty} x_n = x \text{ or } \lim x_n = x \text{ or } x_n \to x$$

for the limit of a sequence.

Example 3-2 :

Let us show that $\lim\limits_{n \to \infty} \frac{n-1}{n} = 1$.

Given $\varepsilon > 0$ as a measure of proximity of $\frac{n-1}{n}$ to 1, if we can find an $N \in \mathbb{N}$ satisfying

the conditions and conclusion of Definition 2-2 then we are done.
For, assume that $\left|\frac{n-1}{n} - 1\right| < \varepsilon$, Then $\left|-\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$; and by Corollary 11-1, there exists an $N \in \mathbb{N}$ such that $\varepsilon > \frac{1}{N}$, for instance $N = [\frac{1}{\varepsilon}] + 1$.
Thus, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\left|\frac{n-1}{n} - 1\right| < \varepsilon$ $\forall n \geq N$.

Exercise 4-2:
Show that the sequence $((-1)^n)$ is divergent.

Theorem 5-2:
If the sequence (x_n) is convergent, its limit is unique.

Definition 6-2:
A sequence (x_n) is bounded if there is a positive number K such that
$$|x_n| < K, \quad \forall n \in \mathbb{N}.$$

Theorem 7-2:
If a sequence is convergent, then it is bounded.

Remark 8-2:
The converse is not correct. For example the sequence $(-1)^n$ is bounded, but it is not convergent.
We will see later on, however, that a bounded sequence has always a convergent subsequence.

Theorem 9-2:
If $x_n \longrightarrow x$ and $x \neq 0$, then there is a positive number M and an integer N such that
$|x_n| > M$ for all $n \geqslant N$.
(i.e. (x_n) is bounded away from zero).

Theorem 10-2:
Let (x_n) and (y_n) be two convergent sequences with limits x and y respectively. Then
i) $(x_n + y_n)$ is convergent to $x+y$.
ii) $(x_n y_n)$ converges to xy.
iii) $\frac{x_n}{y_n}$ converges to $\frac{x}{y}$, provided $y \neq 0$.
iv) if $x_n \leqslant y_n$, $\forall n \in \mathbb{N}$, then $x \leqslant y$.

(but $x_n < y_n \ \forall n \in \mathbb{N} \not\Longrightarrow x < y$)

Theorem 11-2:
Suppose the sequences (x_n), (y_n) and (z_n) satisfy $x_n \leqslant y_n \leqslant z_n \ \forall n \geqslant N$. If $\lim x_n = \lim z_n = l$ then (y_n) is convergent and its limit is l.

Example 12-2:
Show that if $x_n \longrightarrow x$, then $|x_n| \longrightarrow |x|$.
Solution:
For any reals a, b, we have
$$0 \leqslant \big||a| - |b|\big| \leqslant |a - b|.$$
In particular
$$0 \leqslant \big||x_n| - |x|\big| \leqslant |x_n - x|, \ \forall n \in \mathbb{N}.$$
Since $x_n \longrightarrow x$, we see by the squeezing theorem 11-2 that $|x_n| - |x| \longrightarrow 0$.
In other words $|x_n| \longrightarrow |x|$.

Monotonic Sequences:

Definition 13_2:

A sequence (x_n) is said to be

1) increasing if $x_{n+1} \geqslant x_n$, $\forall n \in \mathbb{N}$.
2) strictly increasing if $x_{n+1} > x_n$, $\forall n \in \mathbb{N}$.
3) decreasing if $x_{n+1} \leqslant x_n$, $\forall n \in \mathbb{N}$.
4) Strictly decreasing if $x_{n+1} < x_n$, $\forall n \in \mathbb{N}$

An increasing or decreasing sequence is called a monotonic sequence.

Theorem 14_2:

A monotonic sequence is convergent if and only if it is bounded. More precisely

i) if (x_n) is increasing and bounded above then $\lim x_n = \sup\{x_n, n \in \mathbb{N}\}$

ii) if (x_n) is decreasing and bounded below, then $\lim x_n = \inf\{x_n, n \in \mathbb{N}\}$

Example 15_2:

Show that the sequence $\left(1+\frac{1}{n}\right)^n$ is convergent and calculate its limit.

Solution: put $a_n = \left(1+\frac{1}{n}\right)^n$ and and use the binomial rule

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{1}{n^n}$$ to get

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}$

$$a_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$
$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).$$

and

$$a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots$$
$$+ \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$
$$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right).$$

Comparing x_{n+1} and x_n, we observe that (a_n) is an increasing sequence

on the other hand

$$a_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$
$$= 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 1 + \frac{1}{1-\frac{1}{2}} = 3.$$

Thus a_n is bounded above. So, by Theorem 14-2 the sequence (a_n) is convergent. The limit can be calculated by using the function $f(x) = \left(1+\frac{1}{x}\right)^x$ whose limit is $e^1 = e$; whence

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Cauchy criterion :-

Definition 16-2 :-
The sequence (x_n) is called a Cauchy sequence if, for every $\varepsilon > 0$, there is a positive integer N such that

$$m, n \geq N \Rightarrow |x_n - x_m| < \varepsilon.$$

Definition 17-2 :-
i) A point $x \in \mathbb{R}$ is called a cluster point (or an accumulation point, or a limit point) of a set $A \subset \mathbb{R}$ if every neighborhood V of x contains an element $a \in A$ different from x. The symbol $\hat{A}$ will denote the set of cluster points of A.

ii) A point in A which is not a cluster point of A is called an isolated point of A.

Example 18-2 :-
1) The set of cluster points of any open interval (a, b) is the closed interval $[a, b]$.

2) $\hat{\mathbb{Z}} = \phi$

3) $\hat{\mathbb{Q}} = \hat{\mathbb{Q}^c} = \mathbb{R}$.

Theorem 19-2 :- (Cantor)
Let (I_n) be a sequence of non-empty, closed and bounded intervals.
If $I_{n+1} \subset I_n$ for every $n \in \mathbb{N}$, then the intersection $\bigcap_{n=1}^{\infty} I_n$ is not empty.
Furthermore, if $\inf\{l(I_n), n \in \mathbb{N}\} = 0$ then $\bigcap_{n=1}^{\infty} I_n$ consists of a single point.

uses the completeness assumption.

Theorem 20-2: (Bolzano-Weierstrass)
Every infinite and bounded subset of $\mathbb{R}$ has at least one cluster point in $\mathbb{R}$.
proof: See the textbook p. 93.

Theorem 21-2: (Cauchy's Criterion)
A sequence of real numbers is convergent if and only if it is a Cauchy sequence.
proof:
See the textbook, Theorems 3.8 and 3.9 as well as Lemma 3.1.

Example 22-2:
Consider the real sequence

$$\begin{cases} a_1 = 1, \; a_2 = 2 \\ a_n = \frac{1}{2}(a_{n-1} + a_{n-2}), \; n > 2. \end{cases}$$

1) Show that (x_n) is convergent.
2) Can you calculate its limit.

Solution:
1) By induction, we see that

$$x_n - x_{n+1} = \frac{(-1)^n}{2^{n-1}}, \; \forall n \in \mathbb{N}.$$

Hence if $m > n$, then

$$|x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \dots + |x_{m-1} - x_m|$$
$$= \sum_{r=n}^{m-1} \frac{1}{2^{r-1}}$$

$$= \frac{1}{2^{n-1}} \sum_{r=0}^{m-n-1} \frac{1}{2^r} = \frac{1}{2^{n-1}} \frac{1-\left(\frac{1}{2}\right)^{m-n}}{1-\frac{1}{2}}$$

$$< \frac{1}{2^{n-1}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n-2}}.$$

So, given $\varepsilon > 0$, we can choose N so that $\frac{1}{2^{N-2}} < \varepsilon$, and thereby conclude that

$$m > n \geqslant N \Rightarrow |x_n - x_m| < \frac{1}{2^{n-2}} \leqslant \frac{1}{2^{N-2}} < \varepsilon.$$

Thus (x_n) is a Cauchy sequence.

By Cauchy's Criterion (Thm 21.2), it is convergent.

2) observe that Cauchy's criterion only guarantees the existence of the limit but no value of it is provided. However, we can use induction to get

$$x_{2n+1} = 1 + \sum_{k=1}^{n} \frac{1}{2^{2k-1}}$$

$$= 1 + 2 \sum_{k=1}^{n} \frac{1}{4^k}$$

$$= 1 + 2 \cdot \frac{1}{4} \frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}$$

$$= 1 + \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^n\right) \longrightarrow \frac{5}{3}.$$

on the other hand

$$x_{2n} = x_{2n+1} + \frac{1}{2^{2n-1}} \longrightarrow \frac{5}{3}.$$

So $x_n \longrightarrow \frac{5}{3}$.

Subsequences :

Definition 23-2 :

Let $(x_n, n \in \mathbb{N})$ be a sequence of real numbers. If (n_k) is a strictly increasing sequence of natural numbers :

$$n_1 < n_2 < \dots < n_k < \dots$$

then the sequence

$$(x_{n_k}, k \in \mathbb{N}) = (x_{n_1}, x_{n_2}, x_{n_3}, \dots)$$

is called a subsequence of (x_n).

Example 24-2 :

1) The sequence $(x_{k+2})_{k \in \mathbb{N}}$, i.e. $(x_3, x_4, \dots)$ is a subsequence of the sequence $(x_k)_{k \in \mathbb{N}^*}$

2) The sequences $\left(\frac{1}{2k}\right)$ and $\left(\frac{1}{2k+1}\right)$ are subsequences of the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}^*}$

3) The sequence $(0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$ is not a subsequence of $\left(\frac{1}{n}\right)_{n \in \mathbb{N}^*}$

Theorem 25-2 :

If the sequence (x_n) converges to x, then every subsequence of (x_n) also converges to x.

Corollary 26-2 :

If the sequence (x_n) is convergent and has a subsequence which converges to x, then (x_n) itself converges to the same limit x.

Theorem 27-2 : (See Remark 8-2)

Every bounded sequence has a convergent subsequence.

Theorem 28-2 :

Let (x_n) be a bounded sequence. If every convergent subsequence of (x_n) has the same limit, then (x_n) converges to that same limit.

Upper and lower limits 29-2 :

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence, and define th upper and lower limits of (x_n) by

$$\lim \sup x_n = \inf_{n\in\mathbb{N}} \{ \sup \{ x_k, k \geq n \} \}$$

$$\lim \inf x_n = \sup_{n\in\mathbb{N}} \{ \inf \{ x_k, k \geq n \} \}$$

If (x_n) is not bounded above, then we write

$$\lim \sup x_n = \infty$$

and if it is not bounded below, then we write

$$\lim \inf x_n = -\infty.$$

Remark 30-2 :

observe that

$$\lim \sup x_n \geq \lim \inf x_n$$

$$\lim \sup (-x_n) = - \lim \inf x_n.$$

Example 31-2 :

calcullate the upper and lower limits of the sequence

$$x_n = (-1)^n + \frac{1}{n}.$$

solution :

$$\sup\{x_k, k \geq n\} = \begin{cases} 1+\frac{1}{n}, & n \text{ even} \\ 1+\frac{1}{n+1}, & n \text{ odd.} \end{cases}$$

$$\limsup x_n = \inf_{n\in\mathbb{N}}\{\sup\{x_k, k\geq n\}\}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1.$$

on the other hand

$$\inf\{x_k, k\geq n\} = -1, \text{ since } \frac{1}{n} > 0$$

whence

$$\liminf x_n = \sup_{n\in\mathbb{N}}\{\inf\{x_k, k\geq n\}\}$$

$$= \lim_{n\to\infty}(-1) = -1.$$

Of course the sequence (x_n) is divergent since $(-1)^n$ is divergent and $(\frac{1}{n})$ is convergent

Theorem 32-2 :

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence, and denote the upper and lower limits respectively by

$$x^* = \limsup x_n \quad \text{and} \quad x_* = \liminf x_n.$$

1) Given $\varepsilon > 0$, there is a positive integer N such that

$$x_n < x^* + \varepsilon, \ \forall n \geq N,$$

i.e. $x_n \in (-\infty, x^* + \varepsilon)$ for all but a finite number of $n \in \mathbb{N}$

2) Given $\varepsilon > 0$, and any $m \in \mathbb{N}$, there is an integer $n > m$ such that

$$x_n > x^* - \varepsilon$$

i.e. $x_n \in (x^* - \varepsilon, \infty)$ for infinitely many $n \in \mathbb{N}$.

3) (x_n) Converges to x if and only if $x^* = x_* = x$.

4) There is a subsequence of (x_n) which converges to x^*, and a subsequence of (x_n) which converges to x_*.

5) If a subsequence of (x_n) converges to x, then $x_* \leq x \leq x^*$; that is x^* is the greatest limit that can be attained by a convergent subsequence of (x_n) and x_* is the least such limit.

Chapter III : "Series of numbers"

Definition 1.3 :
Let (x_n) be a sequence of real numbers, and form "the sequence of partial sums"

$$S_1 = x_1$$
$$S_2 = x_1 + x_2$$
$$S_3 = x_1 + x_2 + x_3$$

$$S_n = x_1 + x_2 + \dots + x_n = \sum_{k=1}^{n} x_k.$$

The infinite series generated by (x_n) is the sequence (S_n) of partial sums.
If the sequence of partial sums (S_n) is convergent we say that the series is convergent and its sum is the limit of (S_n), i.e.:

$$\lim_{n \to \infty} S_n = S = \sum_{k=1}^{\infty} x_k,$$

otherwise it is divergent.

Example 2.3 :
let $a \in \mathbb{R}$ and consider the sequence $(a^n)_{n \in \mathbb{N}}$
The general term of the sequence of partial sums is:

$$S_n = 1 + a + a^2 + \dots + a^n = \begin{cases} \dfrac{1 - a^{n+1}}{1 - a}, & a \neq 1 \\ n + 1, & a = 1 \end{cases}$$

If $a = 1$, then $\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} (n+1) = \infty$ and thus the series is divergent.

If $a \neq 1$, then $\lim_{n \to \infty} S_n$ exists if and only if $\lim_{n \to \infty} a^n$ exists, that is if and only if $|a| < 1$, in which case $\lim a^n = 0$, whence

$$\sum_{k=0}^{\infty} a^k = \lim_{n \to \infty} S_n = \frac{1}{1-a}, \text{ (convergent)}.$$

Theorem 3_3 :-
Suppose that Σx_n and Σy_n are convergent series and that $c \in \mathbb{R}$. Then the series

$$\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$$

and

$$\sum_{n=1}^{\infty} c x_n = c \sum_{n=1}^{\infty} x_n$$

are both convergent.

Theorem 4_3 :- (Cauchy's criterion)
The series Σx_n is convergent if and only if, for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that

$$n > m \geq N \Longrightarrow |S_n - S_m| = |x_{m+1} + \cdots + x_n| < \varepsilon.$$

proof :- Exercise (apply Theorem 21_2 with of course Definition 16_2).

Corollary 5_3 :-
If the series Σx_n is convergent, then

$$\lim x_n = 0.$$

proof :- Exercise.

Example 6-3:
The series $\sum \frac{1}{n}$ is known as the harmonic series.
Although $\lim\limits_{n \to \infty} \frac{1}{n} = 0$, this series is not convergent which shows that this condition is only necessary.
We have

$$S_{2n} - S_n = \sum_{k=n+1}^{2n} \frac{1}{k}$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

$$\geq \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n} = \frac{1}{2}$$

$\Rightarrow (S_n)$ is not Cauchy and so it is not convergent by Cauchy's criterion.

Since the sequence is increasing we should have $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Definition 7-3:
The series $\sum x_n$ is said to be absolutely convergent if the series $\sum |x_n|$ is convergent.

Theorem 8-3:
An absolutely convergent series is convergent.

Definition 9-3:
The series $\sum x_n$ is said to be conditionally convergent if $\sum x_n$ is convergent and $\sum |x_n|$ is divergent.

Example 10-3:
1) The series $\sum \frac{(-1)^n}{n^2}$ is absolutely Convergent and hence it is Convergent.
2) The series $\sum \frac{(-1)^n}{n}$ is conditionally Convergent since it is convergent while $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is divergent.

Convergence tests:
Theorem 11-3: (Comparison tests)
If (x_n) and (y_n) are positive sequences that satisfy the inequality $x_n \leq y_n$ for all $n \geq N$, for some fixed positive integer N, then the Convergence of $\sum y_n$ implies the convergence of $\sum x_n$.

Corollary 12-3:
If $0 \leq x_n \leq y_m$, $\forall n \geq N$ and $\sum x_n$ is divergent, then $\sum y_n$ is divergent.

Theorem 13-3: (limit Comparison test)
Suppose (x_n) and (y_n) are positive sequences such that $\lim \frac{x_n}{y_n}$ exists.
i) If $\lim \frac{x_n}{y_n} \neq 0$, the series $\sum x_n$ and $\sum y_n$ either both Converge or they both diverge.
ii) If $\lim \frac{x_n}{y_n} = 0$ and the series $\sum y_n$ Converges, then $\sum x_n$ also converges.

Example 14-3 :
Since $\frac{1}{n} \leqslant \frac{1}{n^p}$, $\forall p \leqslant 1$, $\forall n \in \mathbb{N}$, and since we know that $\sum \frac{1}{n}$ is divergent, we infer that the series $\sum \frac{1}{n^p}$ is divergent for $p \leqslant 1$.

Example 15-3 : (The telescoping series)
The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and sums to 1.
Indeed, we have $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, whence

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$
$$= 1 - \frac{1}{n+1}$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1$

Example 16-3 :
The p-series $\sum \frac{1}{n^p}$ for $p \geqslant 2$ is convergent.
Indeed, just observe that

$$\lim_{n \to \infty} \frac{1}{n(n+1)} \Big/ \frac{1}{n^2} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1 \neq 0$$

whence $\sum \frac{1}{n^2}$ is convergent by the limit comparison test of Theorem 13-3 and by Example 15-3;
on the other hand, $\frac{1}{n^p} \leqslant \frac{1}{n^2}$ for all $p \geqslant 2$, so by the comparison test, $\sum \frac{1}{n^p}$ is convergent for $p \geqslant 2$.

We can also show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $1 < p < 2$ is also Convergent (use for example the integral test).

In Conclusion

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{Converges for } p > 1 \\ \text{diverges for } p \leq 1. \end{cases}$$

Theorem 17-3 : (The root test)
Let (x_n) be a sequence of real numbers, and set

$$r = \lim_{n \to \infty} \sqrt[n]{|x_n|} = \lim_{n \to \infty} |x_n|^{\frac{1}{n}}$$

whenever this limit exists. Then, the series $\sum_{n=1}^{\infty} x_n$ is

1) absolutely convergent if $r < 1$,
2) divergent if $r > 1$,

The test is inconclusive (fails) if $r = 1$.

Example 18-3 :
The series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is Convergent by the root test, since

$$\lim_{n \to \infty} \sqrt[n]{\frac{1}{(\log n)^n}} = \lim_{n \to \infty} \frac{1}{\log n} = 0 < 1.$$

Exercise 19-3 :
Test the Convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Theorem 20-3: (Ratio test)

If the limit $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lambda$, $\left(\begin{matrix}(x_n)\subset\mathbb{R} \text{ is a}\\ \text{sequence}\end{matrix}\right)$

exists, then

1) the series $\sum x_n$ is absolutely convergent if $\lambda<1$
2) the series $\sum x_n$ is divergent if $\lambda>1$.
3) the test fails if $\lambda=1$.

Example 21-3:

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$.

We have $\lim_{n\to\infty}\left|\frac{3^{\cancel{n}+1}(n+1)!}{(n+1)^{n+\cancel{1}}}\cdot\frac{n^n}{3^{\cancel{n}}\,\cancel{n!}}\right| =$

$$= \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^n = \frac{3}{e} > 1$$

Thus the series is divergent.

Exercise 22-3:

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$.

Definition 23-3:

The sequence (x_n) is said to be alternating if the sign of x_n is different from that of x_{n+1} for every n. The generated series is called alternating series.

Theorem 24-3: (alternating series test)

Let (x_n) be a positive decreasing sequence whose limit is 0. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent.

Example 25-3:
Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
We know that the sequence $(\frac{1}{n})$ satisfies the following conditions:

1) $\frac{1}{n} > 0$, $\forall n$ (positive)
2) $\lim_{n \to \infty} \frac{1}{n} = 0$
3) Decreasing since $\frac{1}{n+1} < \frac{1}{n}$, $\forall n$

So by the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent. Actually it is conditionally convergent since $\sum \left|\frac{(-1)^n}{n}\right|$ is divergent, (See Example 10-3).

Theorem 26-3: (Integral test)
Let f be a positive, decreasing, continuous function on $\{t, t \geq 1\}$. Then the series $\sum f(n)$ converges if and only if the improper integral

$$\int_1^{\infty} f(t)\,dt = \lim_{n \to \infty} \int_1^{n} f(t)\,dt$$

exists.

Example 27-3:
Test the convergence of the series $\sum_{n=2}^{\infty} \frac{\log n}{n}$

Consider the function $f(t) = \frac{\log t}{t}$, $t \geq e$.

$f'(t) = \frac{1 - \log t}{t^2} \leq 0$ for $t \geq e$, and thus it is decreasing. Since it is positive and continuous and $f(n) = \frac{\log n}{n}$, the integral test applies.

We have $\int_4^\infty \frac{\log t}{t} dt = \lim_{n \to \infty} \left[(\log t)^2 \cdot \frac{1}{2} \right]_4^n$

$$= \lim_{n \to \infty} \frac{1}{2} \left(\log^2 n - \log^2 4 \right) = \infty.$$

Thus, the series is divergent.

Exercise 28-31

Test the convergence of each one of the series

1) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

2) $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^2}$

Exercise :

1) Suppose that f is increasing on $[1,\infty)$. Show that

$$f(1)+\dots+f(n-1) < \int_1^n f(x)\,dx < f(2)+\dots+f(n).$$

2) Now choose $f = \log$ and show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

3) Deduce that $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

Solution :

1) Since f is increasing we have

$f(i) < f(x) < f(i+1)$

$\Rightarrow f(i)\int_i^{i+1} < \int_i^{i+1} f < f(i+1)\int_i^{i+1}$

$$f(i) < \int_i^{i+1} f(x)\,dx < f(i+1)$$

Summing up from $i=1$ to $n-1$, we get the desired inequality.

2) from (1) we have

$$\log 1+\dots+\log(n-1) < \int_1^n \log x\,dx < \log 2+\dots+\log n$$

$$\log(n-1)! < \left[x\log x - x\right]_1^n < \log n!$$

$$\Rightarrow \log(n-1)! < n\log n - n + 1 < \log n!$$

$$\Rightarrow (n-1)! < \frac{n^n}{e^{n-1}} < n!$$

$$\Rightarrow \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n} < \frac{n^{n+1}}{e^{n-1}}$$

$$\Rightarrow \underset{e \leftarrow}{\frac{1}{e^{1-\frac{1}{n}}}} < \frac{\sqrt[n]{n!}}{n} < \frac{n^{\frac{1}{n}}}{e^{1-\frac{1}{n}}} \qquad n^{\frac{1}{n}} \to 1, \quad e^{1-\frac{1}{n}} \to e$$

Chapter IV : Limits of Functions.

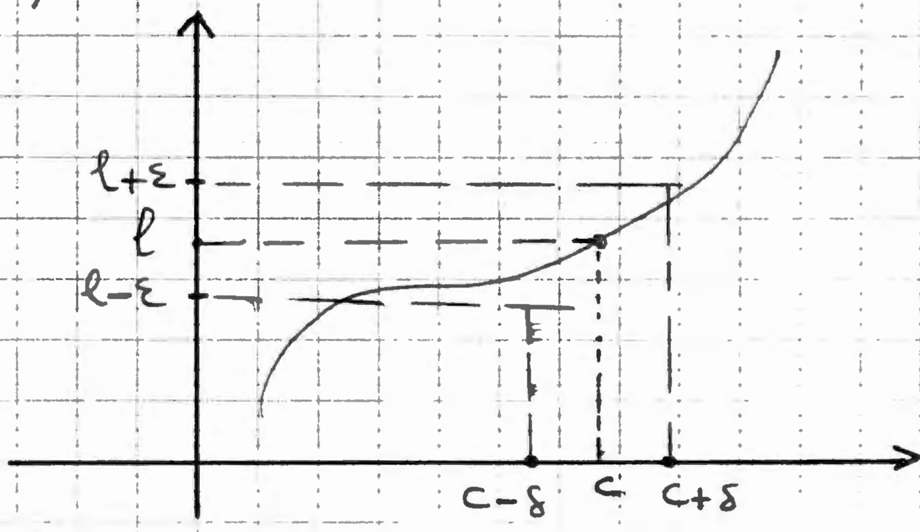
Definition 1-4 :
Let $f : D \longrightarrow \mathbb{R}$ and $c \in \overline{D}$. We say that the limit of f at c is ℓ if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x \in D,\ 0 < |x - c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon,$$

and denote it by

$$\lim_{x \to c} f(x) = \ell$$

or $f(x) \longrightarrow \ell$ as $x \longrightarrow c$.



Example 2-4 :
Show that $\lim_{x \to 2} (3x - 5) = 1$.

Solution :
Given $\varepsilon > 0$, we look for a $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |(3x - 5) - 1| < \varepsilon$$

Now, $|(3x-5) - 1| = 3|x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{3}$
Thus, choose $\delta = \frac{\varepsilon}{3}$, or even any $\frac{\varepsilon}{3} \leq \delta$.

Theorem 3-4:

Let $f: D \longrightarrow \mathbb{R}$ and $c \in \overline{D}$. The following statements are equivalent:

1) $\lim_{x \to c} f(x) = \ell$

2) For every sequence (x_n) in D such that $x_n \neq c$ for any $n \in \mathbb{N}$ and $x_n \longrightarrow c$, the sequence $(f(x_n))$ converges to ℓ.

Example 4-4:

1) The function $f: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{1}{x^2}$$

has no limit at 0, since $x_n = \frac{1}{n} \neq 0$ for any n and $x_n \longrightarrow 0$ but $(f(x_n)) = (n^2)$ diverges, so no limit

2) The function Sgn is defined on $\mathbb{R}$ by

$$\operatorname{Sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

has no limit at 0, because there are sequences $x_n = \frac{1}{n} \neq 0, \forall n$ and $x_n \longrightarrow 0$ and $y_n = -\frac{1}{n} \neq 0, \forall n$ and $y_n \longrightarrow 0$, but $\operatorname{Sgn} x_n \longrightarrow 1$ while $\operatorname{Sgn} y_n \longrightarrow -1$. So the limit does not exist at 0.

Remark 5-4:
The limit if it exists, it must be unique.

proposition 6-4:
Let $f: D \longrightarrow \mathbb{R}$ and $c \in \bar{D}$. If f has a limit at c, then f is bounded near c, i.e.

$$|f(x)| \leq M, \ \forall x \text{ close to } c, \text{ with } M > 0.$$

proposition 7-4:
Let $f, g : D \longrightarrow \mathbb{R}$ and $c \in \bar{D}$. If $\lim\limits_{x \to c} f(x) = l$ and $\lim\limits_{x \to c} g(x) = k$, then

1) $\lim\limits_{x \to c} (f(x) + g(x)) = l + k$
2) $\lim\limits_{x \to c} (f(x) \cdot g(x)) = l k$
3) $\lim\limits_{x \to c} f(x)/g(x) = l/k$, provided $k \neq 0$
4) If $f(x) \leq g(x)$, then $l = \lim\limits_{x \to c} f(x) \leq k = \lim\limits_{x \to c} g(x)$

Theorem 8-4: (Squeezing theorem)
Let $f, g, h : D \longrightarrow \mathbb{R}$ and $c \in \bar{D}$. If

$$f(x) \leq g(x) \leq h(x) \quad \text{near } c$$

and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = l$, then

$$\lim_{x \to c} g(x) = l.$$

Example 9-4:
1) Show that $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

2) show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Solution:

1) We know that, for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, one has

$$\sin|\theta| \leq |\theta| \leq \tan|\theta|$$

So $$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

passing to the limit as $\theta \to 0$, we obtain by the squeezing theorem 8-4 that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

2) Since $|\sin \frac{1}{x}| \leq 1$, $\forall x \neq 0$.

So $$0 \leq |x \sin \frac{1}{x}| \leq |x|$$

Thus $\lim x \sin \frac{1}{x} = 0$ by Thm 8-4.

Definition 10-4: (Right and left limits)

$\lim_{x \to c^+} f(x) = l \iff (\forall \varepsilon > 0, \exists \delta > 0$ s.th $0 < x - c < \delta \Rightarrow |f(x) - l| < \varepsilon)$.

$\lim_{x \to c^-} f(x) = l \iff (\forall \varepsilon > 0, \exists \delta > 0$ s.th. $0 < c - x < \delta \Rightarrow |f(x) - l| < \varepsilon)$.

Lemma 11-4:

The limit of f at c exists if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$.

In which case $\lim_{x \to c} = l = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$

Example 12-4:
Consider the function $f(x)=\begin{cases}\dfrac{x-4}{\sqrt{x}-2} & , x>4\\ 0 & , x=4\\ 2x-4 & , x<4\end{cases}$

Then $\lim\limits_{x\to 4^+} f(x)=\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}=\lim\limits_{x\to 4}(\sqrt{x}+2)=4$

$$\lim_{x\to 4^-} f(x)=\lim_{x\to 4} 2x-4 = 4$$

Since $\lim\limits_{x\to 4^+} f(x)=4=\lim\limits_{x\to 4^-} f(x)$, the limit at $x=4$ exists and equals 4, i.e. $\lim\limits_{x\to 4} f(x)=4$

Definition 13-4:
1) Let $f: D\to\mathbb{R}$ and $c\in\overline{D}$. The limit of f at c is said to be ∞, and write

$$\lim_{x\to c} f(x)=\infty,$$

if given $M\in\mathbb{R}$, there is a $\delta>0$ such that

$$x\in D,\ 0<|x-c|<\delta \Rightarrow f(x)>M.$$

Similarly $\lim\limits_{x\to c} f(x)=-\infty$, if given $M\in\mathbb{R}$ there is $\delta>0$ such that

$$x\in D,\ 0<|x-c|<0 \Rightarrow f(x)<M.$$

2) Let $f: D\to\mathbb{R}$, where $\infty\in\overline{D}$. The limit of f at ∞ is ℓ, and write

$$\lim_{x\to\infty} f(x)=\ell,$$

if, given $\varepsilon>0$, there is an M such that

$$x\in D,\ x>M \Rightarrow |f(x)-\ell|<\varepsilon$$

Similarly, if $-\infty\in\overline{D}$, we have

$$\lim_{x\to-\infty} f(x)=m$$

if given $\varepsilon>0$, there is $M\in\mathbb{R}$ such that

Example 14 - 4 :-
Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution :-
Given any $M > 0$. Then, we look for $\delta > 0$ such that $x \in D$, $|x - 0| < \delta \Rightarrow \frac{1}{x^2} > M$

$$\frac{1}{x^2} > M \Rightarrow x^2 < \frac{1}{M} \Rightarrow |x| < \frac{1}{\sqrt{M}}.$$

So take δ such that $0 < \delta \leq \frac{1}{\sqrt{M}}$.

Monotonic functions :-
A function is said to be monotonic if it is either increasing or decreasing.

Theorem 15 - 4 :-
Let (a, b) be any open interval in $\mathbb{R}$. If the function $f : (a, b) \to \mathbb{R}$ is monotonic, the set of points $A \subseteq (a, b)$ where f has no limit is countable. Furthermore, for every $c \in (a, b) \setminus A$, we have

$$\lim_{x \to c} f(x) = f(c).$$

Exercise :-

prove that the sequence

$$x_1 = 1 \;,\; x_{n+1} = \sqrt{3+x_n} \;,\; \forall n \in \mathbb{N}$$

is convergent and determine its limit :-

Solution ;

Let us show that (x_n) is bounded above

We have $x_1 \leq 1 \leq 3$

suppose $x_n \leq 3 \Rightarrow 3 + x_n \leq 6$

$$\Rightarrow x_{n+1} = \sqrt{3+x_n} \leq \sqrt{6} \leq 3 .$$

So $x_n \leq 3 \;,\; \forall n \in \mathbb{N}$.

Let us show that (x_n) is increasing :-

$x_1 = 1 \;,\; x_2 = \sqrt{3+1} = 2 > x_1$

suppose $x_n > x_{n-1}$ and show $x_{n+1} > x_n$.

$$\Rightarrow x_n > x_{n-1} \Rightarrow 3 + x_n > 3 + x_{n-1}$$

$$\Rightarrow x_{n+1} = \sqrt{3+x_n} > x_n = \sqrt{3+x_{n-1}}$$

$\Rightarrow (x_n)$ is increasing.

Now, (x_n) is increasing and bounded above, so it is convergent.

Let us calculate its limit :

suppose the limit is x, so

$$x_{n+1} = \sqrt{3+x_n} \Rightarrow \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{x_n+3}$$

$$\Rightarrow x = \sqrt{3+x} \Rightarrow x^2 = 3 + x \Rightarrow x^2 - x - 3 = 0$$

$$\Delta = (-1)^2 - 4(-3)(1) = 13 \Rightarrow x = \frac{1 \mp \sqrt{13}}{2}$$

$\Rightarrow$ the limit is $x = \frac{1+\sqrt{13}}{2} \approx 2.3$

Chapter V : Continuity

Definition 1-5:

Let $f\ D \longrightarrow \mathbb{R}$ and $c \in D$. The function f is said to be continuous at c if if given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x \in D,\ |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon.$$

Theorem 2-5:

A function $f: D \longrightarrow \mathbb{R}$ is continuous at $c \in D$ if and only if, for every sequence (x_n) in D which converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Example 3-5

Let f be defined on $(-1, 1)$ and satisfy

$$|f(x)| \leqslant |x|, \text{ for all } x \in (-1, 1).$$

Show that f is continuous at $x = 0$.

Solution:

First, observe that $0 \in (-1, 1)$, whence

$$|f(0)| \leqslant |0| = 0 \Longrightarrow f(0) = 0$$

Thus we want to show $\lim\limits_{x \to 0} f(x) = 0$.

Given $\varepsilon > 0$, look for $\delta > 0$ such that

$|f(x) - 0| < \varepsilon$ whenever $|x - 0| < \delta$.

so

$$|f(x) - 0| = |f(x)| \leqslant |x| = |x - 0| < \delta < \varepsilon$$

Thus, choose $\delta = \varepsilon$, to see that

$\forall\ \varepsilon > 0,\ \exists\ \delta = \varepsilon > 0$ such that $|x| < \delta \Longrightarrow$

$$|f(x)| < \varepsilon.$$

Theorem 4-5:

1) If the functions $f: D \longrightarrow \mathbb{R}$ and $g: D \longrightarrow \mathbb{R}$ are continuous at $c \in D$, then

i) $f+g$ and $f \cdot g$ are both continuous at c.

ii) f/g is continuous at c provided $g(c) \neq 0$.

2) Let $f: D \longrightarrow \mathbb{R}$, $g: E \longrightarrow \mathbb{R}$, and $f(D) \subseteq E$. If f is continuous at $c \in D$ and g is continuous at $f(c)$, then $g \circ f: D \longrightarrow \mathbb{R}$ is continuous at c.

Definition 5-5

Let $f: D \longrightarrow \mathbb{R}$ be a function.

1) f is said to be bounded on $E \subseteq D$ if $\exists\, M \geqslant 0$ such that $|f(x)| \leqslant M \;\forall x \in E$.

f is said to be bounded if it is bounded on D.

2) f is said to have a minimum on D if there is a point $x_1 \in D$ such that

$$f(x) \geqslant f(x_1), \;\forall x \in D.$$

3) f is said to have a maximum on D if $\exists\, x_2 \in D$ such that $f(x) \leqslant f(x_2)$, $\forall x \in D$.

Theorem 6-5:

If I is a closed and bounded set, and the function $f: I \longrightarrow \mathbb{R}$ is continuous, then f has a maximum and a minimum on I i.e. $\sup\limits_{I} f(x)$ and $\inf\limits_{I} f(x)$ exists.

(and $\sup\limits_{I} f(x) = \max\limits_{I} f(x)$, $\inf\limits_{I} f(x) = \min\limits_{I} f(x)$)

Theorem 7-5: (Intermediate value property)
Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. If λ is a real number between $f(a)$ and $f(b)$, then there is a point $c \in (a, b)$ such that $f(c) = \lambda$.

Corollary 8-5:
1) Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ is continuous, then $f(I)$ is an interval.
If $I = [a, b]$ is bounded and closed, then $f([a, b])$ is also closed and bounded.

2) If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then there is $c \in (a, b)$ such that $f(c) = 0$.

Theorem 9-5:
Let $f : I \longrightarrow \mathbb{R}$, where I is an interval, be continuous and injective. Then
1) f is strictly monotonic.
2) f^{-1} is continuous and strictly monotonic

Example 10-5:
If $f : [0, 1] \longrightarrow [0, 1]$ is a continuous function. Show that it has a fixed point, that is a point $x_0 \in [0, 1]$, where $f(x_0) = x_0$.
Solution:
Consider the function $g(x) = f(x) - x$ on $[0, 1]$. We will show that g has a zero in $[0, 1]$.
If $f(0) = 0$ or $f(1) = 1$, we can take x_0 to be either 0 or 1.

If $f(0) \neq 0$ and $f(1) \neq 1$, then

$$g(0) = f(0) > 0 \text{ and } g(1) = f(1) - 1 < 0$$

because $f([0,1]) \subseteq [0,1]$.

$g(x) = f(x) - x$ is continuous on $[0,1]$. So by Corollary 8-5, there is an $x_0 \in (0,1)$ such that $g(x_0) = 0$, i.e. $f(x_0) = x_0$.

Exercise 11-5:
Let the function $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and satisfy the condition: for every $x \in [a,b]$ there is a $y \in [a,b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there is a $c \in [a,b]$ with $f(c) = 0$.

Definition 12-5: (uniform Continuity)

A function $f: D \longrightarrow \mathbb{R}$ is said to be uniformly continuous on $E \subset D$ if, for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$x, t \in E, \ |x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon.$$

Remark 13-5:
1) uniform continuity is on sets, not points
2) uniformly continuous $\Longrightarrow$ continuous.

Theorem 14-5:
The function $f: D \longrightarrow \mathbb{R}$ is uniformly continuous if and only if, for every pair of sequences (x_n) and (y_n) in D satisfying $|x_n - y_n| \longrightarrow 0$, we have $|f(x_n) - f(y_n)| \longrightarrow 0$.

continuity on compact sets 15-5:
For any set $E \subset \mathbb{R}$, the following statements are equivalent:
1) E is said to be compact.
2) E is closed and bounded.
3) Every sequence in E has a subsequence which converges to a point in E.

Theorem 16-5:
If f is continuous on the compact set $D \subseteq \mathbb{R}$, then f is uniformly continuous, and $f(D)$ is compact in $\mathbb{R}$.

Chapter VI : "Differentiation"

Definition 1-6 :
Let $f : I \longrightarrow \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval and $c \in I$. f is said to be differentiable at c if the following limit

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists.
The value $f'(c)$ is called the derivative of f at the point c.
If f is differentiable at every $x \in I$, we say that f is differentiable, and its derivative function is $f' : I \longrightarrow \mathbb{R}$.
For closed intervals $[a, b]$, we write

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

and

$$f'(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

If $y = f(x)$, then we write $\frac{dy}{dx}\Big/_{x=c}$ for $f'(c)$.
Also $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Example 2-6 :
Calculate the derivative of $f(x) = |x|$ if exists.
Solution :
We would have $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

$$= \lim_{x \to c} \frac{|x| - |c|}{x - c}.$$

Let (x_n) be a sequence converging to c.
If $c > 0$, $\exists\ N_1 \in \mathbb{N}$ such that $x_n > 0$, $\forall\ n \geqslant N_1$.
and thus $\frac{|x_n| - |c|}{x_n - c} = \frac{x_n - c}{x_n - c} = 1$, $\forall\ n \geqslant N_1$

whence $f'(c) = 1$.

If $c<0$, $\exists N_2 \in \mathbb{N}$ such that $x_n<0, \forall n \geqslant N_2$

whence $\dfrac{|x_n|-|c|}{x_n-c} = \dfrac{-x_n+c}{x_n-c} = -1, \forall n \geqslant N_2$

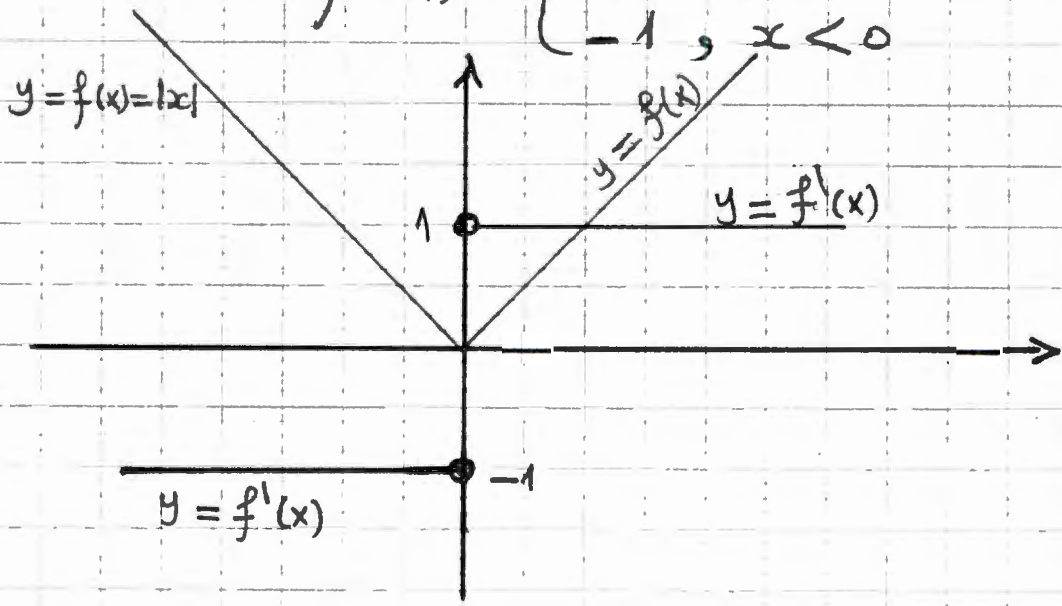
$$\Rightarrow f'(c) = -1$$

If $c=0$, choose $x_n = \frac{(-1)^n}{n} \longrightarrow 0$ to see that

$$\frac{|x_n|-0}{x_n-0} = \frac{\frac{1}{n}}{\frac{(-1)^n}{n}} = (-1)^n \text{ divergent.}$$

So $f'(0)$ does not exist.

Hence:
$$f'(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$



Theorem 3-6:
If $f: I \longrightarrow \mathbb{R}$ is differentiable at $c \in I$, then it is continuous at c.

proof:
If $x \neq c$, then $f(x) - f(c) = \dfrac{f(x)-f(c)}{x-c}(x-c)$

$$\Rightarrow \lim_{x\to c}(f(x)-f(c)) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \lim_{x\to c}(x-c)$$
$$= f'(c) \cdot 0 = 0.$$

Theorem 4-6:
Suppose f, g are differentiable at $c \in I$. Then
1) $(f+g)'(c) = f'(c) + g'(c)$.
2) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
3) if $g(c) \neq 0$, $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$.
4) $(f^n)'(c) = n f^{n-1}(c) f'(c)$.

Theorem 5-6:
f differentiable on I, g differentiable on $J \supseteq f(I)$
then $g \circ f$ is differentiable on I and

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

i.e. $(g \circ f)' = (g' \circ f) \cdot f'$.

Similarly, if $y = f(x)$ and $\omega = g(y)$, then

$$\frac{d\omega}{dx} = \frac{d\omega}{dy}\frac{dy}{dx}.$$

Theorem 6-6:
Let f be injective and continuous on I. If f is differentiable at $b \in I$, then f^{-1} is also differentiable at $c = f(b)$ if and only if $f'(b) \neq 0$, in which case $(f^{-1})'(c) = \frac{1}{f'(b)}$.

Example 7-6:
If $f(x) = \sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, find $(f^{-1})'(x)$.

Solution:

f is injective and differentiable on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Since $f'(x) = \cos x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, the inverse function $f^{-1}: [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is differentiable on $(-1, 1)$. Its derivative is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}}$$

$$= \frac{1}{\sqrt{1-x^2}}.$$

Thus $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$.

The mean value Theorem:

Definition 8-6:

The function $f: D \longrightarrow \mathbb{R}$ is said to have a local maximum at $c \in D$ if there is a neighborhood $U = (c-\delta, c+\delta)$ of c such that

$$f(x) \leq f(c), \forall x \in U \cap D$$

f has a local minimum at $c \in D$ if there is a neighborhood $U = (c-\delta, c+\delta)$ such that

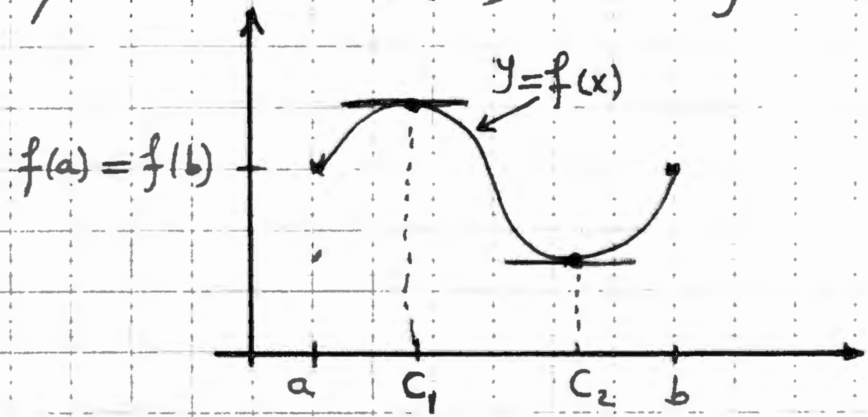
$$f(x) \geq f(c), \forall x \in U \cap D.$$

We say that f has an extremum if either it has a minimum or a maximum

Theorem 9-6:

If f has an extremum at $c \in (a, b)$, and if f is differentiable at c, then $f'(c) = 0$.

Theorem 10-6: "Rolle's Theorem"
If f is continuous on $[a, b]$, differentiable on (a, b), and satisfies $f(a)=f(b)$, then there is a point $c \in (a, b)$ where $f'(c)=0$.



Theorem 11-6: "Mean value Theorem"
If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ s.th.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Corollary 12-6:
Let f, g be continuous functions on $[a, b]$ and differentiable on (a, b).
1) If $f'(x) = g'(x)$, $\forall x \in (a, b)$, then there is a constant c such that $f(x) = g(x) + c$, $\forall x \in [a, b]$
In particular, if $f'(x)=0$, $\forall x \in (a, b)$, then f is constant on $[a, b]$.
2) If for all $x \in (a, b)$, $f'(x) \neq 0$, then f is injective on $[a, b]$.

Theorem 13-6:
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function which is differentiable on (a, b).
i) If $f'(x) \geq 0$ $\forall x \in (a, b)$, then f is increasing on $[a, b]$
ii) If $f'(x) > 0$, $\forall x \in (a, b)$, then f is strictly increasing on $[a, b]$

iii) If $f'(x) \leq 0, \forall x \in (a,b)$, then f is decreasing on $[a,b]$.

iv) If $f'(x) < 0, \forall x \in (a,b)$, then f is strictly decreasing.

Example 14-6 :-

Show that $-x \leq \sin x \leq x \; , \forall x \geq 0$.

Solution :

The function $\sin t$, $t \in [0,x]$, $x > 0$, is continuous and differentiable on $(0,x)$. So by the M.V.Th. $\exists\, c \in (0,x)$ with

$$\sin x - \sin 0 = (x-0)\cos c$$

$$\Rightarrow \quad \sin x = x \cos c \Rightarrow |\sin x| = |x||\cos c|$$

But $|\cos c| \leq 1$, whence

$$|\sin x| \leq |x| \; , \forall x > 0 .$$

Also $|\sin(-x)| \leq |-x| \Leftrightarrow |-\sin x| \leq |-x|$

i.e. $|\sin x| \leq |x|$ even if $x < 0$.

Thus $|\sin x| \leq |x|$, $\forall x \in \mathbb{R}$, which is written also as

$$-x \leq \sin x \leq x \; , \forall x \in \mathbb{R} .$$

Theorem 15-6 :- "first derivative test"

Let c be a critical point and a point of continuity for the function $f : D \longrightarrow \mathbb{R}$.

1) If there is an open interval $I \subseteq D$, which contains c, such that

$$f'(x) < 0 \text{ for all } x \in I, x < c$$

$$f'(x) > 0 \text{ for all } x \in I, x > c$$

then $f(c)$ is a local minimum for f.

ii) If there is an open interval $I \subset D$, which contains c such that

$$f'(x) > 0 \;,\; \forall\, x \in I \;,\; x < c$$

$$f'(x) < 0 \;,\; \forall\, x \in I \;,\; x > c$$

then $f(c)$ is a local maximum for f.

iii) If there is an open interval $I \subseteq D$, which contains c such that $f'(x)$ has the same sign on $I \setminus \{c\}$, then $f(c)$ is not a local extremum for f.

Theorem 16-6 :- (Darboux)
Let the function $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable. If λ is a number between $f'(a)$ and $f'(b)$, then there is a point $c \in (a,b)$ such that $f'(c) = \lambda$.

Example 17-6 :
Determine the local extrema of the function

$$f(x) = x + \frac{1}{x} \;,\; x \neq 0.$$

Solution :
f is differentiable on $(-\infty, 0)$ and on $(0, \infty)$

$$f'(x) = 1 - \frac{1}{x^2} = 0 \implies x = \pm 1.$$

x	$-\infty$	-2	-1	0	1	2	∞
$f'(x)$		$\frac{3}{4}$	0	$\Vert$	0	$3/4$	
$f(x)$	$-\infty$ $\nearrow$		$\searrow$	$-\infty$ $\Vert$ ∞	$\searrow$ $\nearrow$		∞

$f(-1) = -2$ a local maximum, $f(1) = 2$ a local minimum

Theorem 17-6 :- (Cauchy's mean value theorem)
If the functions f and g are continous on $[a, b]$ and differentiable on (a, b), then there is a point $c \in (a, b)$ such that

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

proof :-
Consider $h(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$.
and apply Rolle's theorem on $[a, b]$.

Theorem 19-6 :- (L'Hopital's rule)
Let f, g be continous on $I \subseteq \mathbb{R}$ and differentiable on $I \setminus \{c\}$, where $c \in I$. If

i) $g'(x) \neq 0$, $\forall x \in I \setminus \{c\}$.

ii) $f(c) = g(c) = 0$.

iii) $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$,

then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Corollary 20-6 :-
Let f, g be differentiable on (a, b) and $g'(x) \neq 0$ for any $x \in (a, b)$. If $\lim\limits_{x \to a^+} g(x) = \lim\limits_{x \to a^+} f(x) = \infty$ and $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)}$ exists in $\overline{\mathbb{R}}$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Corollary 21-6:
Let f, g be differentiable on $[a, \infty)$ and suppose

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0,$$

$$g'(x) \neq 0, \forall x > a.$$

If $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists in $\overline{\mathbb{R}}$, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

Example 22-6:
Calculate the following limits

1) $\lim_{x \to \infty} \frac{\cos x - 1}{x^2}$

2) $\lim_{x \to 0^+} \left(1 + \frac{2}{x}\right)^x$

3) $\lim_{x \to \infty} \frac{\log x}{\sqrt{x}}$

Solution:

1) $\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = \lim_{x \to 0} -\frac{\cos x}{2} = -\frac{1}{2}$

2) $\lim_{x \to 0^+} \left(1 + \frac{2}{x}\right)^x = \lim_{x \to 0^+} e^{x \ln\left(1 + \frac{2}{x}\right)}$

$$= \lim_{x \to 0^+} e^{\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}} = \lim_{x \to 0^+} e^{\frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}}$$

$$= \lim_{x \to 0^+} e^{\frac{+2}{1+\frac{2}{x}}} = e^0 = 1.$$

3) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$

Exercise 23-6: (on the mean value theorem)
Show that if $f: (a, b) \longrightarrow \mathbb{R}$, (or $f: \mathbb{R} \longrightarrow \mathbb{R}$)

uniformly
continuous

is differentiable and f' is bounded, then f is uniformly continuous on (a, b).

Solution:

Let $x, y \in (a, b)$ be arbitrary, and apply the mean value theorem on $[x, y]$.
Then there is $z \in (x, y)$ such that

$$f'(z) = \frac{f(y) - f(x)}{y - x}$$

$$\Rightarrow |f(y) - f(x)| = |y - x|\,|f'(z)|.$$

But f' is bounded, i.e. $\exists\, M > 0$ such that

$$|f'(t)| \leq M, \quad \forall\, t \in (a, b).$$

Thus $|f(x) - f(y)| \leq M|x - y|, \ \forall x, y \in (a,b)$

Given $\varepsilon > 0$, there is $\delta = \frac{\varepsilon}{M} > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq M|x - y| < \cancel{M}\frac{\varepsilon}{\cancel{M}} = \varepsilon$$

i.e. f is uniformly continuous on (a, b).

Theorem 24-6: (Taylor's Theorem)
Let the function f and its derivatives $f', f'', \ldots$ $\ldots f^{(n)}$ be continuous on $[a, b]$ and suppose $f^{(n)}$ is differentiable on (a, b). If x_0 is a point in $[a, b]$ then, for every $x \in [a, b] \setminus \{x_0\}$, there is a point c between x_0 and x such that

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots$$
$$\ldots + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1}.$$

$n = 0$, we recover the M.V. Th.

Example 25-6 :
Approximate the number e to 6 decimal places.
solution :-
Consider $f(x) = e^x$ with $x_0 = 0$.
Taylor's formula yields

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^c$$

$0 < c < x$.

For $x = 1$ and the margin of error 10^{-6}, we have

$$\frac{e^c}{(n+1)!} \leqslant 10^{-6} \quad , \quad 0 < c < 1$$

Since $e^c < e < 3$, we see that $(n+1)! \geqslant 3 \times 10^{6}$

i.e. $n \geqslant 9$.

For $n = 9$, we have

$$e \approx 1 + 1 + \frac{1}{2} + \dots + \frac{1}{9!} = 2.718282$$

with an error of 10^{-6}.

Theorem 26-6 (Taylor - Young)
Let the function f and all its derivatives up to the order n be continuous on $[a, b]$, and suppose $f^{(n)}$ is differentiable at the point $x_0 \in [a, b]$.
If $x \in [a, b]$ then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$+ \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} + E_n(x)$$

where $E_n(x)/(x - x_0)^{n+1} \longrightarrow 0$ as $x \longrightarrow x_0$

corollary 27-6 : (second derivative test)

Suppose $f'(c) = f''(c) = \dots = f^{(m-1)}(c) = 0$ and $f^{(m)}(c) \neq 0$. Then

i) For $f(c)$ to be an extremum of f, m has to be an even number.

ii) When m is even, $f(c)$ is a local maximum if $f^{(m)}(c) < 0$, and a local minimum if $f^{(m)}(c) > 0$.

Example 28-6 :

Decide whether $f(0)$ is a local extremum of f if $f(x) = \sin x + \cos x - x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Solution :

$f'(x) = \cos x - \sin x - 1 + x + \frac{1}{2}x^2 \longrightarrow f'(0) = 1 - 0 - 1 + 0 + 0 = 0$

$f''(x) = -\sin x - \cos x + 1 + x \longrightarrow f''(0) = 0 - 1 + 1 + 0 = 0$

$f'''(x) = -\cos x + \sin x + 1 \longrightarrow f'''(0) = -1 + 0 + 1 = 0$

$f''''(x) = \sin x - \cos x \longrightarrow f''''(0) = 0 - 1 = -1 \neq 0$

$\Longrightarrow f(0)$ is an etremum.

Now $f''''(0) = -1 < 0 \Longrightarrow f(0) = 1$ is a local maximum.

Chapter VII : "Riemann Integral"

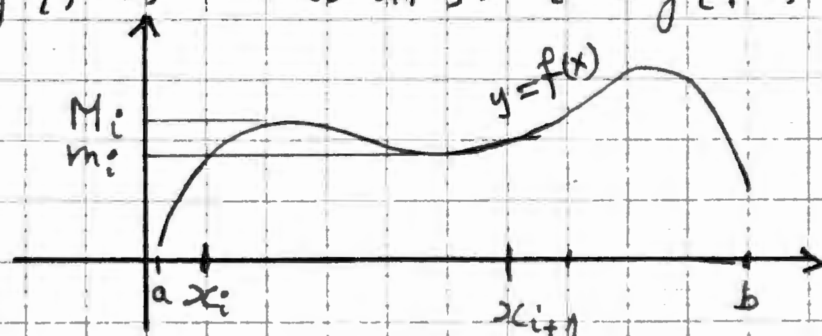
Let $[a, b]$ be a closed and bounded interval in $\mathbb{R}$, and let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded real-valued function on $[a, b]$.

Definition 1-7 :
A finite ordered set of points $P = \{x_0, x_1, \dots, x_n\}$ is called a partition of $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

For $i = 0, 1, \dots, n-1$, set

$$M_i = \operatorname{Sup}\{f(x),\ x \in [x_i, x_{i+1}]\},\quad m_i = \operatorname{Inf}\{f(x),\ x \in [x_i, x_{i+1}]\}$$



Definition 2-7 :
The upper sum $U(f, P)$ and lower sum $L(f, P)$ of f with respect to P are defined by

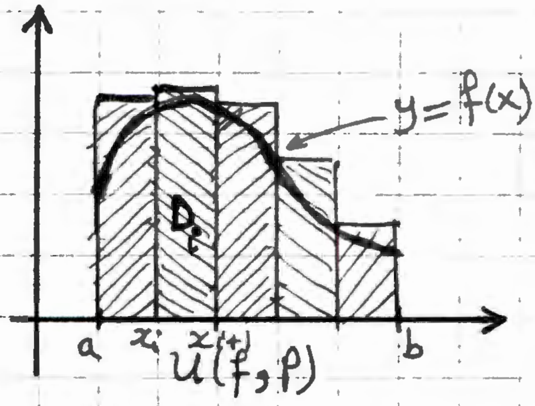
$$U(f, P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i).$$

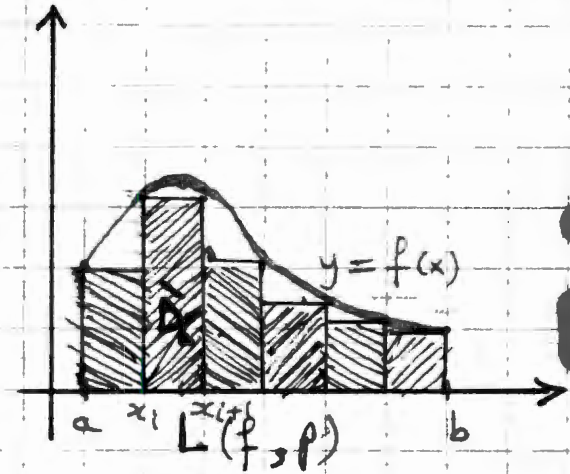
$$L(f, P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i).$$

and observe that always we have

$$L(f, P) \leqslant U(f, P).$$

\# observe that if $f \geqslant 0$ on $[a, b]$, then $U(f, P)$ is the area of the union of rectangles D_i, where the base of D_i is $[x_i, x_{i+1}]$ and its height is M_i. Likewise $L(f, P)$ is the area of the union of rectangles $\grave{D}_i$ whose base is $[x_i, x_{i+1}]$ and whose height is m_i:





Definition 3-7:
we say that a partition Q is finer than the partition P if, as sets, $P \subseteq Q$.

For example $P \cup Q$ is a new partition which is finer that both of P and Q.

If Q is finer than P, then

$U(f, Q) \leqslant U(f, P)$ and $L(f, Q) \geqslant L(f, P)$.

Lemma 4-7:
For any partitions P, Q on $[a, b]$, we have

$$U(f, P) \geqslant L(f, Q).$$

\# Denote by $\mathcal{P}(a, b)$ the set of all partitions P of $[a, b]$ and put $A = \{ U(f, P), P \in \mathcal{P}(a, b) \}$ $B = \{ L(f, P), P \in \mathcal{P}(a, b) \}$ and observe that $\operatorname{Inf} A$ and $\operatorname{Sup} B$ exist.

Definition 5-7:
The upper integral $\mathcal{U}(f)$ and the lower integral $L(f)$ of f over $[a,b]$ are defined as

$$\mathcal{U}(f) = \inf A = \inf\{\mathcal{U}(f,P),\ P \in \mathcal{P}[a,b)\}$$

$$L(f) = \sup B = \sup\{L(f,P),\ P \in \mathcal{P}(a,b)\}$$

and we always have

$$\mathcal{U}(f) \geqslant L(f).$$

Definition 6-7:
let $f: [a,b] \longrightarrow \mathbb{R}$ be bounded. We say that f is Riemann integrable over $[a,b]$ if $\mathcal{U}(f) = L(f)$. In which case the common value is called the Riemann integral of f over $[a,b]$ and is denoted by $I(f)$ or $\int_a^b f$.

often we write $I(f) = \int_a^b f(x)\,dx$, and note that

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du = \cdots$$

The class of Riemann integrable functions over $[a,b]$ is denoted by $R(a,b)$.
when the integral $\int_a^b f(x)\,dx$ exists, we write $f \in R(a,b)$

Example 7-7:
Suppose that $f(x) = c$ constant on $[a,b]$.
If P is any partition, then
$P = \{x_0, x_1, \ldots, x_n\}$, $M_i = m_i = c$, whence

$$\mathcal{U}(f,P) = L(f,P) = \sum_{i=0}^{n-1} c(x_{i+1} - x_i) = c(b-a)$$

Therefore $\mathcal{U}(f) = L(f) = c(b-a)$,
and thus $f \in R(a,b)$ with $\int_a^b c\,dx = c(b-a)$.

Theorem 8-7: (Riemann's criterion)
The following statements are equivalent:

i) $f \in R(a,b)$.

ii) For all $\varepsilon > 0$, there is a partition $P \in \mathcal{P}(a,b)$ such that $U(f,P) - L(f,P) < \varepsilon$.

Corollary 9-7:
$f \in R(a,b)$ if and only if there is a sequence (P_n) in $\mathcal{P}(a,b)$ such that
$$U(f,P_n) - L(f,P_n) \longrightarrow 0$$
in which case
$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

Example 10-7:
Let $f : [a,b] \longrightarrow \mathbb{R}$ be defined by $f(x) = x$.
Show that $f \in R(a,b)$ and calculate $\int_a^b f(x)\,dx$.

Solution:
If $P = \{x_0, x_1, \dots, x_n\}$ is any partition of $[a,b]$, then put $M_i = x_{i+1}$, $m_i = x_i$ for $i = 0, 1, \dots, n-1$.

Thus, we obtain
$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2$$

Choosing P_n to be the uniform partition into n parts, then $x_i = a + i\,\frac{b-a}{n}$, and
$$U(f,P_n) - L(f,P_n) = \sum_{i=0}^{n-1} \frac{(b-a)^2}{n^2} = \frac{(b-a)^2}{n} \xrightarrow[n\to\infty]{} 0.$$

Thus $f \in R(a,b)$.

Furthermore, we have

$$L(f,P_n) = \sum_{i=0}^{n-1}\left(a + i\,\frac{b-a}{n}\right)\frac{b-a}{n} = a(b-a) + \frac{(b-a)^2}{n^2}\,\frac{n(n-1)}{2}$$

Thus $\lim\limits_{n\to\infty} L(f,P_n) = \frac{1}{2}(b^2 - a^2)$.

Hence $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$.

Theorem 11-7:

1) If f is monotonic on $[a,b]$, then $f \in R(a,b)$.

2) If f is continuous on $[a,b]$, then $f \in R(a,b)$.

Remark 12-7:

The converse is not correct, for instance the function $f(x) = \begin{cases} 0, & x \in [0,1]\setminus Q \\ \frac{1}{q}, & x = p/q \in [0,1]\cap Q \end{cases}$

is integrable, (i.e. $f \in R(a,b)$) with

$\int_0^1 f(x)\,dx = 0$, while f is not continuous.

Definition 13-7:

If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$ then $\|P\| = \max\{x_{i+1} - x_i,\ i = 0, 1, \dots, n-1\}$ is called the norm of the partition P.

Theorem 14-7: (Darboux theorem)

Let $f: [a,b] \longrightarrow \mathbb{R}$ be bounded. Given $\varepsilon > 0$, there is a $\delta > 0$ such that, if P is any partition of $[a,b]$ satisfying $\|P\| < \delta$, then

$U(f,P) - U(f) < \varepsilon$ and $L(f) - L(f,P) < \varepsilon$.

$\lim U(f,P) = U(f)$ and $\lim L(f,P) = L(f)$.

Definition 15-7:
Let $P=\{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.
We say that $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ is a mark on P if $\alpha_i \in [x_i, x_{i+1}]$, $\forall\, i \in \{0, 1, \dots, n-1\}$.
We then call the sum

$$S(f,P,\alpha) = \sum_{i=0}^{n-1} f(\alpha_i)(x_{i+1}-x_i)$$

the Riemann sum of f on P with mark α.
We always have

$$L(f,P) \leq S(f,P,\alpha) \leq U(f,P)$$

Theorem 16-7:
The following are equivalent:

i) $f \in \mathcal{R}(a,b)$ with integral over $[a,b]$ equal to A.

ii) For any $\varepsilon > 0$, there is a $\delta > 0$ such that, if P is any partition satisfying $\|P\| < \delta$ and α is any mark on P, then $|S(f,P,\alpha) - A| < \varepsilon$

i.e. $\displaystyle\lim_{\|P\| \to 0} S(f,P,\alpha) = A$.

Corollary 17-7:
Suppose that $f \in \mathcal{R}(a,b)$. If (P_n) is a sequence of partitions such that $\|P_n\| \to 0$, then, for any choice of marks α_n on P_n, we have

$$\int_a^b f = \lim_{n \to \infty} S(f, P_n, \alpha_n).$$

Example 18-7:
Using Riemann sums, evaluate $\int_0^1 (x-x^2)\,dx$.
solution:
Since $f(x) = x - x^2$ is continuous, it is integrable by Theorem 11-7.
Let $P_n = \{x_0, x_1, \dots, x_n\}$ be the uniform partition of $[0,1]$. Then $x_i = \frac{i}{n}$, $i = 0, 1, \dots, n$.
choosing $\alpha_{n,i} = x_{i-1}$, so $\alpha_n = (x_0, x_1, \dots, x_{n-1})$
we obtain

$$S(f, P_n, \alpha_n) = \sum_{i=0}^{n-1} \left(\frac{i}{n} - \frac{i^2}{n^2}\right)\frac{1}{n}$$
$$= \frac{1}{n^2}\sum_{i=0}^{n-1} i - \frac{1}{n^3}\sum_{i=0}^{n-1} i^2$$
$$= \frac{1}{n^2}\,\frac{n(n-1)}{2} - \frac{1}{n^3}\,\frac{(n-1)\,n(2n-1)}{6}$$

Since $\|P_n\| = \frac{1}{n} \longrightarrow 0$, we obtain

$$\int_0^1 (x-x^2)\,dx = \lim_{n\to\infty} S(f, P_n, \alpha_n)$$
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

19-7. properties of the integral:

1) $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$.

2) $f \geqslant 0$, $f \in R(a,b) \Longrightarrow \int_a^b f \geqslant 0$.

3) $f \leqslant g \Longrightarrow \int_a^b f \leqslant \int_a^b g$.

4) $\int_a^b f = \int_a^c f + \int_c^b f$, $c \in (a,b)$.

5) $f \geqslant 0$ and continuous, then $\int_a^b f = 0 \Longleftrightarrow f = 0$ on $[a,b]$

6) $\int_a^b f = -\int_b^a f$.

7) $\int_a^a f = 0$

Theorem 20-7: (Mean value Theorem)
If f is continuous on $[a,b]$, then there is a point $c \in (a,b)$ such that
$$\int_a^b f(x)\,dx = f(c)(b-a).$$

proposition 21-7:
1) If $f: [a,b] \longrightarrow [c,d]$ is Riemann integrable, and $\varphi: [c,d] \longrightarrow \mathbb{R}$ is continuous, then $\varphi \circ f$ is Riemann integrable on $[a,b]$.

2) If $f \in R(a,b)$, then $|f| \in R(a,b)$ and
$$\left|\int_a^b f\right| \leq \int_a^b |f|$$
The converse is not true.

3) If $f \in R(a,b)$, then $f^n \in R(a,b)$ for all $n \in \mathbb{N}$.

4) If $f, g \in R(a,b)$, then $f.g \in R(a,b)$.

5) If f is continuous on $[a,b]$ and
$$F(x) = \int_a^x f(t)\,dt \quad , x \in [a,b],$$
then F is differentiable on $[a,b]$ and $F' = f$.

Now we have the fundamental theorem of calculus:

Theorem 22-7: (Fundamental Theorem of Calculus)
If F is differentiable on $[a, b]$ and $F' \in R(a,b)$
then
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$
$$= \left[F(x)\right]_a^b$$

Theorem 23-7: (Substitution rules)
1) Suppose φ is differentiable on $[a, b]$ and its derivative φ' is continuous. If f is continuous on the range of φ, then
$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx.$$
2) Let $\varphi : [a, b] \longrightarrow \mathbb{R}$ have a continuous derivative that does not vanish anywhere in (a, b).
If f is continuous on the range of φ, and ψ is the inverse of φ, then
$$\int_a^b f(\varphi(t))\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,\psi'(x)\,dx.$$

Theorem 24-7: (Integration by parts)
Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be differentiable on $[a, b]$. If $f', g' \in R(a, b)$, then
$$\int_a^b f(x)\,g'(x)\,dx = \left[f(x)\,g(x)\right]_a^b - \int_a^b f'(x)\,g(x)\,dx$$
$$= f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'(x)\,g(x)\,dx.$$

Example 25-7:
Suppose that f is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = a$. Show that

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt = a.$$

Solution:
Let N be large enough and apply the M.V. Th. for f on $[N, x]$, then
$\exists\, c_x \in [N, x]$, such that

$$\int_N^x f(t)\,dt = (x-N)\, f(c_x).$$

$$\Longrightarrow \frac{1}{x}\int_N^0 f(t)\,dt + \frac{1}{x}\int_0^x f(t)\,dt = \left(1 + \frac{N}{x}\right) f(c_x)$$

passing to the limit as $x \to \infty$, we get

$$\underbrace{\lim_{x\to\infty} \frac{1}{x}\int_N^0 f(t)\,dt}_{\to 0} + \lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)\,dt = \lim_{x\to\infty}\Big(1+\underbrace{\frac{N}{x}}_{\to 0}\Big)\underbrace{f(c_x)}_{\to a}$$

Thus $\displaystyle\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt = a$, since N arbitrary.

Example 26-7:
Let f be continuous on $[0,\infty)$ and suppose that it does not vanish on $(0,\infty)$. If

$(f(x))^2 = \int_0^x f(t)\,dt$, $\forall x > 0$, show that $f(x) = \frac{x}{2}$, $x \in [0,\infty)$.

Example 27-7:
Evaluate $I = \displaystyle\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx.$

Improper integrals :-

1) unbounded integrands:

a) $\int_{a^+}^{b} f(x)\,dx$:-

If f is integrable on $(a,b]$ and $\lim_{x\to a^+} f(x)=\pm\infty$ then $\int_{a^+}^{b} f(x)\,dx$ is improper and can be defined as follows :- $\int_{a^+}^{b} f(x)\,dx = \lim_{t\to a^+}\int_{t}^{b} f(x)\,dx.$

If this limit exists, we say that the improper integral $\int_{a^+}^{b} f(x)\,dx$ is convergent and its value equals that limit, i.e.

$$\int_{a}^{b} f(x)\,dx = \int_{a^+}^{b} f(x)\,dx = \lim_{t\to a^+}\int_{t}^{b} f(x)\,dx$$

If the limit does not exist or equals $\pm\infty$ we say that the improper integral $\int_{a^+}^{b} f(x)\,dx$ is divergent

b) $\int_{a}^{b^-} f(x)\,dx$:-

In the same manner, we define it through the limit

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{b^-} f(x)\,dx = \lim_{s\to b^-}\int_{a}^{s} f(x)\,dx.$$

c) $\int_{a}^{b} f(x)\,dx$ where there is a $c\in(a,b)$ with $\lim_{x\to c} f(x)=\pm\infty$:

If f is unbounded near c, we divide the integral as follows

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

The we test the integrals

$$\int_a^{c^-} f(x)\,dx \quad \text{and} \quad \int_{c^+}^{b} f(x)\,dx$$

if both of them converge, then the improper integral $\int_a^b f(x)dx$ converges and its value equals the sum of their values. If one of them diverges then $\int_a^b f(x)dx$ diverges

we can write

$$\int_a^b f(x)\,dx = \lim_{t_1 \to c^-} \int_a^{t_1} f(x)\,dx + \lim_{t_2 \to c^+} \int_{t_2}^{b} f(x)\,dx.$$

observe that if $\int_a^b f(x)dx$ is convergent then the following limit exists

$$\lim_{\varepsilon \to 0^+} \left\{ \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right\}$$

it is called the Cauchy principal value, but its existence does not imply the convergence of the integral $\int_a^b f(x)dx$.

2) Unbounded intervals :-

a) $\int_a^{\infty} f(x)\,dx$:-

If $f : [a, \infty) \longrightarrow \mathbb{R}$ is Riemann integrable on $[a, c]$, $\forall\ \varepsilon > 0$, and if

$$\lim_{M\to\infty}\int_a^M f(x)\,dx$$

exists, then we say that the improper integral $\int_a^\infty f(x)\,dx$ is convergent, and its value equals the value of that limit; otherwise we say that $\int_a^\infty f(x)\,dx$ is divergent

b) $\int_{-\infty}^b f(x)\,dx$:-

In the same manner, we can define this improper integral :-

$$\int_{-\infty}^b f(x)\,dx = \lim_{m\to-\infty}\int_m^b f(x)\,dx.$$

c) $\int_{-\infty}^\infty f(x)\,dx$:-

Similarly, we test the integrals, ($a \in \mathbb{R}$ arbitrary)

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty}\int_a^M f(x)\,dx$$

and

$$\int_{-\infty}^a f(x)\,dx = \lim_{m\to\infty}\int_m^b f(x)\,dx$$

if both converge, then $\int_{-\infty}^\infty f(x)\,dx$ is convergent

and $\int_{-\infty}^\infty f(x)\,dx = \lim_{m\to-\infty}\int_m^a f(x)\,dx + \lim_{M\to\infty}\int_a^M f(x)\,dx$

and if one of them diverge, then $\int_{-\infty}^\infty f(x)\,dx$ diverg

The value $\lim_{t\to\infty}\left\{\int_{-t}^{a} f(x)\,dx + \int_{a}^{t} f(x)\,dx\right\}$

is called the principal value of Cauchy.

Theorem 28-7:- (comparison test)
Let $f, g \in R(a,t)$ for $t > a$ be such that

$$|f(x)| \leqslant g(x), \quad \forall\, x \in [a,\infty).$$

Then, if the improper integral $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is convergent as well.

Theorem 29-7:- (Limit comparison test)
If f, g are positive continuous functions on $[a,\infty)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L > 0$, then the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are either both convergent, or are both divergent.
(Same is true for $(-\infty, b]$, $[a,b)$ or $(a,b]$).

Example 30-7:
1) Riemann integrals $\int_a^b \frac{dx}{(b-x)^\alpha}$, $-\infty < a < b < \infty$ converge if and only if $\alpha < 1$.
2) Riemann integrals $\int_a^\infty \frac{dx}{x^\alpha}$ converge iff $\alpha > 1$.
3) Bertrand integrals, $\int_a^\infty \frac{dx}{x^\alpha (\ln x)^\beta}$, $a > 0$, $\alpha, \beta \in \mathbb{R}$ converge iff either $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

Examples:- See problems set

Chapter VIII :- "Sequences and series of functions"

A) Sequences of functions

Let $D \subseteq \mathbb{R}$ and suppose that for each $n \in \mathbb{N}$, we have a function $f_n : D \longrightarrow \mathbb{R}$.

Then, the sequence $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of "real-valued" functions

For a fixed $x \in D$, $(f_n(x))_{n \in \mathbb{N}}$ is a sequence numbers. If $(f_n(x))_{n \in \mathbb{N}}$ converges for every $x \in D$, we say that $(f_n)_{n \in \mathbb{N}}$ converges pointwise and

$$f(x) = \lim_{n \to \infty} f_n(x)$$

is the pointwise limit of (f_n), and we write

$\lim f_n = f$ or $f_n \longrightarrow f$ on D.

and we have the definition :-

Definition 1-8 :-

$(f_n)_{n \in \mathbb{N}}$ is said to converge pointwise to f on D if, given $\varepsilon > 0$, there is for each $x \in D$ an $N = N(x, \varepsilon) > 0$ such that

$$\forall\, n \geq N \implies |f_n(x) - f(x)| < \varepsilon$$

Example 2-8 :-

$$f_n : [0,1] \longrightarrow \mathbb{R}$$
$$x \longrightarrow f_n(x) = x^n$$

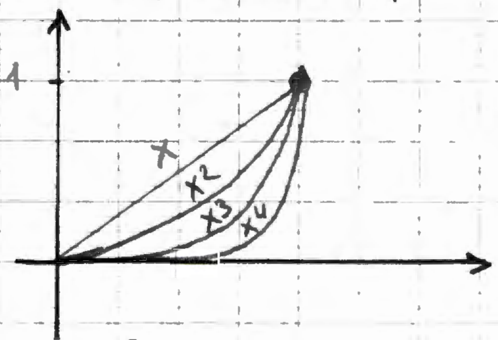
For every $0 \leq x < 1$, we have $\lim\limits_{n \to \infty} f_n(x) = 0$.

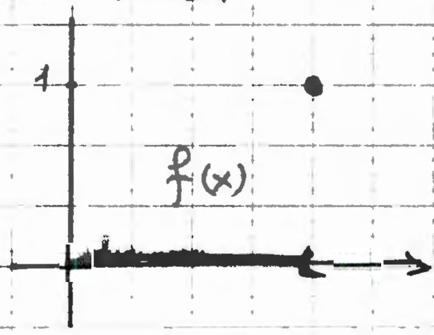
And since $f_n(1) = 1 \longrightarrow 1$, we see that the pointwise limit of is given by

$$f(x) = \begin{cases} 0 , & 0 \leq x < 1 \\ 1 , & x = 1. \end{cases}$$

f_n is continuous for every n, while the limit function f is not.

$$\lim_{x \to 1} \lim_{n \to \infty} f_n(x) \neq \lim_{n \to \infty} \lim_{x \to 1} f_n(x)$$





Example 3-8 :-

$f_n(x) = n x (1 - x^2)^n$ on $[0,1]$.

$f_n(x) \longrightarrow 0$, $\forall x \in (0,1)$ and $f_n(0) = f_n(1) = 0$ for every n. So $\lim\limits_{n \to \infty} f_n(x) = 0$, $\forall x \in [0,1]$.

$f_n \in \mathcal{R}(0,1)$ and $\int_0^1 f_n(x) = \frac{n}{2(n+1)} \xrightarrow[n \to \infty]{} \frac{1}{2}$.

But $\int_0^1 \lim\limits_{n \to \infty} f_n(x)\, dx = \int_0^1 0 . dx = 0$

$$\Rightarrow \lim_{n \to \infty} \int_0^1 f_n(x) dx \neq \int_0^1 \lim_{n \to \infty} f_n(x) dx$$

Example 4-8:-

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow f_n(x) = \begin{cases} -1 & , x < -\frac{1}{n} \\ \sin\left(\frac{n\pi x}{2}\right) & , |x| \leq \frac{1}{n} \\ 1 & , x > \frac{1}{n} \end{cases}$$

So

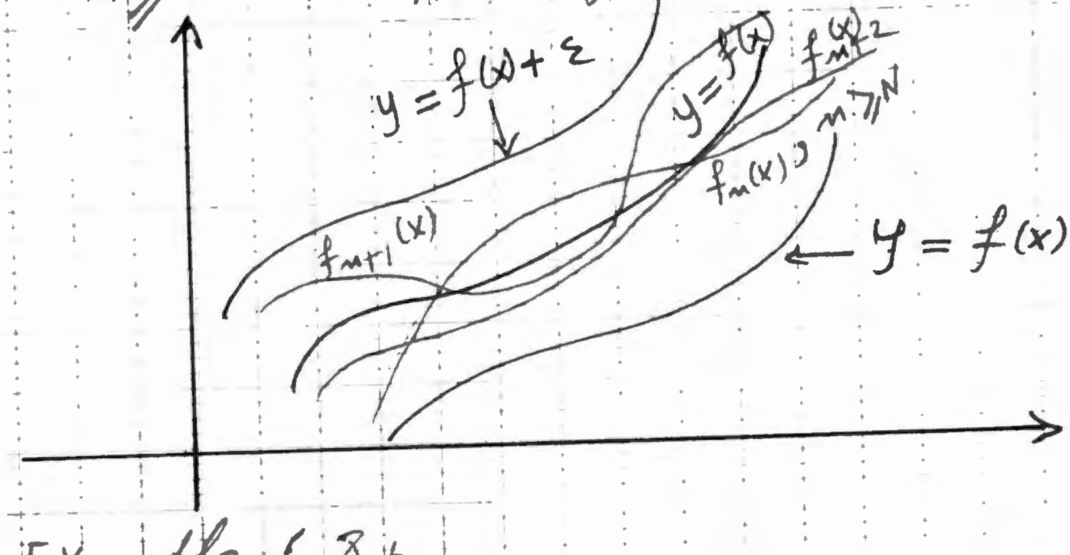
$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} -1 & , x < 0 \\ 0 & , x = 0 \\ 1 & , x > 0 \end{cases}$$

f_n is differentiable, while $f(x) = \text{Sign}(x)$ is not even continuous.

Definition 5-8:- "uniform convergence"

A sequence of functions (f_n) of real functions on $D \subset \mathbb{R}$ is said to converge uniformly to f on D, and denote this by $f_n \xrightarrow{u} f$ if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon) > 0$ such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon, \ \forall x \in D.$$



Example 6-8:-

1) $f_n(x) = \frac{\sin(nx)}{n} \longrightarrow 0$ pointwise on $\mathbb{R}$.

So, given $\varepsilon > 0$, we have

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n}\right| = \frac{1}{n}|\sin nx| \leq \frac{1}{n}, \forall x \in \mathbb{R}$$

So take $\frac{1}{\varepsilon} \leq N \in \mathbb{N}$, and observe that for $n \geqslant N$, we have

$$|f_n(x) - f(x)| = \frac{1}{n}|\sin nx| < \varepsilon, \forall x \in \mathbb{R}.$$

In other words $f_n \xrightarrow{u} f$.

2) $f_n(x) = x^n \longrightarrow 0$ on $(0, 1)$ but

$f_n \not\xrightarrow{u} 0$ (not uniformly)

Remark 6-8:-

$$f_n \xrightarrow{u} f \Longrightarrow f_n \longrightarrow f \text{ pointwise.}$$

while

$$f_n \longrightarrow f \text{ pointwise} \not\Longrightarrow f_n \xrightarrow{u} f.$$

Theorem 7-8:-

$f_n : D \longrightarrow \mathbb{R}$ a sequence of functions. Then $\{f_n\}$ Converges uniformly to f on D if and only if $\sup_{x \in D} |f_n(x) - f(x)| \longrightarrow 0$

Example 8-8:-

Let $\{f_n\}$ be defined by $f_n(x) = \frac{nx}{1+nx}$, $x \geqslant 0$.

1) Find $\lim_{n \to \infty} f_n(x) = f(x)$.

2) Show that if $a > 0$, then $\{f_n\}$ Converges uniformly to f on $[a, \infty)$.

3) Show that (f_n) does not converge uniformly to f on $[0, \infty)$.

solution :-

1) $\lim\limits_{n\to\infty} f_n(0) = f(0) = 0$.

If $x > 0$, then $\lim\limits_{n\to\infty} \frac{nx}{1+nx} = 1$

So $\lim\limits_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \end{cases}$

2) If $x \geq a$, then

$$\left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} \leq \frac{1}{1+na} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \sup_{[a,\infty)} |f_n(x) - f(x)| \leq \frac{1}{1+na} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \{f_n\}$ converges uniformly to $f(x) \equiv 1$ on $[a,\infty)$

3) If $n \in \mathbb{N}$ and $0 < x < \frac{1}{n}$, then

$$\left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} > \frac{1}{2}, \text{ as } nx > 1$$

So $\sup\limits_{[0,\infty)} |f_n(x) - f(x)| \not\longrightarrow 0$ as $n \to \infty$.

$\Rightarrow \{f_n\}$ does not converge uniformly to f on $[0,\infty)$

Theorem 9-8 :- "Cauchy Criterion for unif. con

$\{f_n\}$ is uniformly convergent to f on D if and only if : $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$, $\forall x \in D$

(or $\sup\limits_{D} |f_n(x) - f_m(x)| < \varepsilon$)

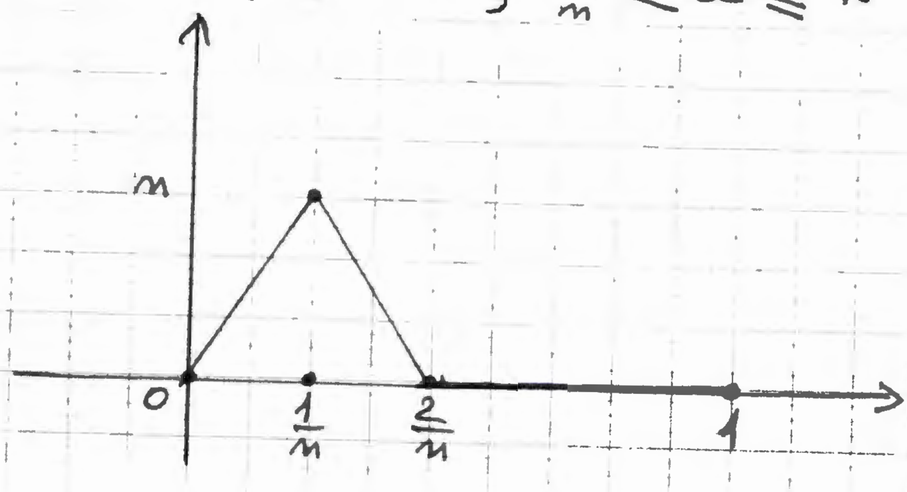
Theorem 10-8 :-
If (f_n) is a sequence of continuous function on D such that $f_n \xrightarrow{u} f$ on D, then f is also continuous on D.

Theorem 11-8 :
If (f_n) is a sequence of functions that are integrable on $[a, b]$ such that $\lim_{n\to\infty} f_n = f$.
If $\{f_n'\}$ converges uniformly on $[a, b]$ then $\{f_n\}$ is also uniformly convergent and $f_n' \longrightarrow f'$.

Example 12-8 :
Consider the sequence of functions on $[0,1]$

$$f_1(x) = 1$$

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x < \frac{1}{n} \\ 2n - n^2 x, & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} < x \leq 1. \end{cases}$$



$n > 2 \Rightarrow \int_0^1 f_n(x)\,dx = \frac{1}{2}\left(\frac{2}{n}\right) \cdot n = 1$, (Area of Triangle $\frac{1}{2}$ B. H.)

$$\Rightarrow \lim_{n \to \infty} \int_0^1 f_n(x)\,dx = 1$$

But $f_n(x) \longrightarrow 0$ on $[0, 1]$

So $\lim_{n \to \infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 \left(\lim_{n \to \infty} f_n(x)\right) dx$.

Theorem 13-8 :-

If $f_n \in R(a, b)$, $\forall n \geq 1$ and $f_n \xrightarrow{u} f$ on $[a, b]$, then

1) $f \in R(a, b)$.

2) $\int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx$.

B) Series of functions :-

Let $\{f_n\}$ be a sequence of functions on $D \subset \mathbb{R}$.

The sum $f_1 + f_2 + f_3 + \cdots + f_n + \cdots$ is called a series of functions and denoted

$$\sum_{n=1}^{\infty} f_n(x).$$

The sequence of partial sums related to it is :-

$$S_n(x) = \sum_{k=1}^{n} f_n(x).$$

If (S_n) is convergent (pointwise) on D, we say

that $\sum_{n=1}^{\infty} f_n(x)$ is convergent pointwise and

$$S(x) = \lim_{n \to \infty} S_n(x) = \sum_{k=1}^{\infty} f_k(x), \quad x \in D.$$

If (S_n) is divergent, then $\sum_{n=1}^{\infty} f_n(x)$ is also diverge

If (S_n) is uniformly convergent, then $\sum_{n=1}^{\infty} f_n(x)$ is also uniformly convergent.

If $\sum |f_n|$ is convergent, then $\sum_{n=1}^{\infty} f_n$ is absolutely convergent.

Theorem 14-8:
If the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $D \setminus \{x\}$, and if $\forall n \geq 1$, $\lim_{t \to x} f_n(t)$ exists, then

$$\lim_{t \to x} \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} \lim_{t \to x} f_n(t).$$

($\Rightarrow$ If f_n continuous $\forall n \Rightarrow \sum f_n$ is continuous)

Theorem 15-8:

$$\left.\begin{array}{l} f_n \in R(a,b) \ \forall n \\ \sum_{n=1}^{\infty} f_n \text{ uniformly convergent on } [a,b] \end{array}\right\} \sum_{n=1}^{\infty} f_n = f \in R(a,b)$$

and $$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

Theorem 16-8:
If f_n is differentiable on $[a,b]$, $\forall n \in \mathbb{N}$ and if $\sum_{n=1}^{\infty} f_n$ is convergent, and if

$\sum_{n=1}^{\infty} f_n'$ is uniformly convergent on $[a,b]$, then $\sum_{n=1}^{\infty} f_n$ is also uniformly convergent and its sum is a differentiable function and $\left(\sum_{n=1}^{\infty} f_n\right)'(x) = \sum_{n=1}^{\infty} f_n'(x)$.

Theorem 17-8: (Cauchy criterion for unif.-Cvg.)
The series $\sum f_n$ is uniformly convergent on D if and only if: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.th. $n, m \geq N \Rightarrow \left|\sum_{k=m+1}^{n} f_k(x)\right| < \varepsilon$, $\forall x \in D$.

Theorem 18-8: (Wierstrass M-test)
Let (f_n) be a sequence of functions on D, and (M_n) a sequence of real numbers such that $|f_n(x)| \leq M_n$, $\forall x \in D$, $n \in \mathbb{N}$.
If $\sum M_n$ is convergent, then both of $\sum f_n$ and $\sum |f_n|$ are uniformly convergent on D.

Example 19-8:-
Test the convergence of the series:-

1) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\mathbb{R}$.
2) $\sum_{n=1}^{\infty} \frac{\cos^2(nx)}{n^2}$.
3) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^{3/2}}$.

Solution :-

1) $\sup_{\mathbb{R}} |S_n(x) - S_{n-1}(x)| = \sup_{\mathbb{R}} |f_n(x)| \geq |f_n(n)| = \frac{n^n}{n!} \geq 1$

so by Cauchy criterion, it is not uniformly convergent on $\mathbb{R}$; while it is convergent pointwise to e^x and $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$.

2) $M_n = \frac{1}{n^2}$ and apply Weierstrass : $\left| \frac{\cos^2(nx)}{n^2} \right| \leq \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so we have uniform convergence on $\mathbb{R}$.

3) It is uniformly convergent on $\mathbb{R}$ since

$\left| \frac{\sin(nx)}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}} < \infty$.

C) Power Series:

A power series is a particular case of series of functions when $f_n(x) = a_n (x-c)^n$, and it takes the form :

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

It is always convergent to a_0 at $x = c$.

without loss of generality, we consider $\sum_{n=0}^{\infty} a_n x^n$.

The radius of convergence is

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

and the interval of convergence take the form :-

$(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$ or $(-\infty, \infty)$.

Theorem 20-8 :-

If R is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, then: the series is absolutely convergent for $|x| < R$, divergent for $|x| > R$, and uniformly convergent on $[-r, r]$ for $0 < r < R$. The case $|x| = R$ should be examined separately ($x = R$ and $x = -R$).