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Chapter 3: Systems of Linear Equations

Def. A "system" of linear equations is a set of linear equations you deal with all together at once.

- Example:

$$\begin{aligned} 2x - 4y &= -4 \\ x + 2y &= 8 \end{aligned}$$

$$\begin{aligned} 3x + 5y &= -3 \\ x - 5y &= -5 \end{aligned}$$

$$\begin{aligned} 2x - 4y &= -4 \\ x + 2y &= 8 \end{aligned}$$

$$\begin{aligned} x + 5y + 2z &= 1 \\ -x - 2y + 7z &= 17 \\ 2x + 8y - 2z &= 6 \end{aligned}$$

$$\begin{aligned} x + 3z &= 4 \\ 2x + y &= -8 \\ 4x + 3y + 3z &= -5 \end{aligned}$$

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$$
\begin{aligned}
2x_1 + x_2 + 3x_3 + 4x_4 &= 5 \\
x_3 + 4x_4 &= 2 \\
4x_1 + 2x_2 + 6x_3 + 8x_4 &= 10 \\
6x_1 + 3x_2 + 14x_3 + 35x_4 &= 33
\end{aligned}
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

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Number of solutions:

One Solution

No Solution

Infinitely Many Solutions

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Cramer’s Rule:

- Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows: $\boldsymbol{AX} = \boldsymbol{B}$, where the $n \times n$ matrix $\boldsymbol{A}$ has a nonzero determinant. Then the theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, \dots, n.$$

where A_i is the matrix formed by replacing the i-th column of $\boldsymbol{A}$ by the column vector $\boldsymbol{B}$.

Case 2 by 2: Given the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

This system has the unique solution:

$$x = \frac{\det(A_1)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_2)}{\det(A)}$$

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \det(A_1) = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad \det(A_2) = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

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Example 1. Solve by Cramer's rule the following linear systems:

$$\begin{cases} 2x+\ y=-1 \\ 4x+3y=1 \end{cases}$$

Sol.

$$\det(A)=\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}=2\cdot 3-4\cdot 1=6-4=2,$$

$$\det(A_1)=\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}=-1\cdot 3-1\cdot 1=-3-1=-4,$$

$$\det(A_2)=\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}=2\cdot 1-4\cdot(-1)=2+4$$

$$x=\frac{-4}{2}=-2, \quad \text{and} \quad y=\frac{6}{2}=3.$$

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Case 3 by 3: Given the system $\begin{array}{l} a_1x+b_1y+c_1x=d_1 \\ a_2x+b_2y+c_2x=d_2 \\ a_3x+b_3y+c_3x=d_3 \end{array}$. This system has the unique solution:

$$x=\frac{\det(A_1)}{\det(A)},\quad y=\frac{\det(A_2)}{\det(A)},\quad \text{and}\quad z=\frac{\det(A_3)}{\det(A)}$$

where

$$\det(A)=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\quad \det(A_1)=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}\quad \det(A_2)=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\quad \det(A_3)=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example 2. Solve by Cramer's rule the following linear systems:

$$\begin{array}{l} 2\cdot x+4\cdot y+3\cdot z=5 \\ 9\cdot x+6\cdot y+8\cdot z=7 \\ 11\cdot x+13\cdot y+10\cdot z=12 \end{array}$$

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Sol.

Write down the coefficient and parameter matrices,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

Calculate the determinant for A (this will be reused), and calculate the determinants for matrices below. Note: when trying to find the first parameter 'x' we replace the first column in A with **B**.

$$x = \frac{\begin{vmatrix} 5 & 4 & 3 \\ 7 & 6 & 8 \\ 12 & 13 & 10 \end{vmatrix}}{|A|} =$$

$$y = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 9 & 7 & 8 \\ 11 & 12 & 10 \end{vmatrix}}{|A|} =$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 5 \\ 9 & 6 & 7 \\ 11 & 13 & 12 \end{vmatrix}}{|A|} =$$

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Gauss Elimination Method:

- Given a linear system $AX = B$.
- Construct the augmented matrix for the system $[AB]$.
- Use elementary row operations to transform the augmented matrix $[AB]$ into a triangular one $[A' B']$.
- Write down the new linear system $A' X = B'$ for which the triangular matrix is the associated augmented matrix.
- Solve the new system by method of back substitution.

- Elementary Row Operations: There are three types of elementary row operations which may be performed on the rows of a matrix:
  - Type 1: R_{ij}, Swap the positions of two rows i and j.
  - Type 2: kR_i, Multiply the row i by a nonzero scalar k.
  - Type 3: kR_i+R_j, Add to the row j a scalar multiple of the row i.

- We notice that these operations do not change the solution set.

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Example: Solve by Gauss Elimination Method the following linear systems:

$$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$$

Sol.

$$[AB]=\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right] \xrightarrow[\frac{3}{2}R_1+R_2]{R_1+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right] \xrightarrow{-4R_2+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

Write down the new linear system associated with the obtained augmented matrix:

$$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$$

Solve the new system by method of back substitution: From the 3rd equation we get: z =-1. Substitute the value of z in the 2nd equation we obtain: 1/2 y - 1/2 = 1, that is, y=3. Substitute the values of z and y in the 1st equation we obtain: 2x +3 - (-1) = 8, that is, x=2. Thus the solution is (2, 3, -1).

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Example. Solve by Gauss Elimination Method the following linear systems:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5. \end{aligned}$$

Sol.

$$[AB] = \begin{pmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{pmatrix} \xrightarrow{2R_1+R_2} \begin{pmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

Write down the new linear system associated with the obtained augmented matrix:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned}$$

Solve the new system by method of back substitution: From the 2nd equation we get: $x_2 = 3$. Substitute the value of x_2 in the 1st equation we obtain: $x_1 + 5(3) = 7$, that is, $x_1 = -8$.

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Example: Solve by Gauss Elimination Method the following linear systems:

$$\begin{aligned} 2x_1 + x_2 - x_3 + 2x_4 &= 5 \\ 4x_1 + 5x_2 - 3x_3 + 6x_4 &= 9 \\ -2x_1 + 5x_2 - 2x_3 + 6x_4 &= 4 \\ 4x_1 + 11x_2 - 4x_3 + 8x_4 &= 2 \end{aligned}$$

Sol.

$$[AB] = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 4 & 5 & -3 & 6 & 9 \\ -2 & 5 & -2 & 6 & 4 \\ 4 & 11 & -4 & 8 & 2 \end{array}\right] \xrightarrow{R_1+R_3, -2R_1+R_4, -2R_1+R_2} \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 6 & -3 & 8 & 9 \\ 0 & 9 & -2 & 4 & -8 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3, -3R_2+R_4} \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right] \xrightarrow{R_3+R_4} \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 5 \\ 0 & 3 & -1 & 2 & -1 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right]$$

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The linear system associated to the new augmented matrix is :

$$\begin{aligned} 2x_1 + x_2 - x_3 + 2x_4 &= 5 \\ 3x_2 - x_3 + 2x_4 &= -1 \\ -x_3 + 4x_4 &= 11 \\ 2x_4 &= 6 \end{aligned}$$

Solving by back substitution, we obtain

$$x_4 = 3, \qquad x_3 = 1, \qquad x_2 = -2, \qquad x_1 = 1.$$

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Gauss-Jordan Elimination Method:

The Gauss-Jordan elimination method to solve a system of linear equations $\boldsymbol{AX} = \boldsymbol{B}$ is described in the following steps:

1. Write the augmented matrix of the system $[\boldsymbol{AB}]$.
2. Use row operations to transform the augmented matrix in the form $[\boldsymbol{A'B'}]$ with $\boldsymbol{A'}$ is the identity matrix. The solution $\boldsymbol{X}$ is the values of the vector $\boldsymbol{B'}$, that is, $\boldsymbol{X} = \boldsymbol{B'}$.

Example: Solve by Gauss-Jordan Elimination Method:

$$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$$

Sol.

$$[AB] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right] \xrightarrow[\frac{3}{2}R_1+R_2]{R_1+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right] \xrightarrow{4R_2+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+\frac{1}{2}R_3]{R_1-R_3} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow[-R_3]{2R_2,} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\frac{1}{2}R_1]{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus the solution is (2, 3, -1).

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More Examples:

Example 2: Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

Example 3: Use Gaussian elimination to solve the system of linear equations

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

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Solutions:

$$\begin{array}{rcrcrcr} & & 2x_2 & + & x_3 & = & -8 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Swap Row 1 and Row 2.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Add Row 1 to Row 3.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ & - & x_2 & - & x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

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Swap Row 2 and Row 3.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & - & x_2 & - & x_3 & = & 3 \\ & & 2x_2 & + & x_3 & = & -8 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{pmatrix}$$

Add twice Row 2 to Row 3.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & - & x_2 & - & x_3 & = & 3 \\ & & & - & x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

Add -1 times Row 3 to Row 2.
Add -3 times Row 3 to Row 1.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & 6 \\ & - & x_2 & & & = & 5 \\ & & & - & x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & -2 & 0 & 6 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

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Add -2 times Row 2 to Row 1.

$$\begin{array}{rcl} x_1 \qquad\qquad\qquad & = & -4 \\ -\; x_2 \qquad\qquad & = & 5 \\ -\; x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

Multiply Rows 2 and 3 by -1.

$$\begin{array}{rcl} x_1 \qquad\qquad & = & -4 \\ x_2 \qquad & = & -5 \\ x_3 & = & 2 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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$$\begin{cases} x+y+z=5 \\ 2x+3y+5z=8 \\ 4x+5z=2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x=3, \quad y=4, \quad z=-2.}$$

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