Chapter 4: Methods of integrals

Section 4.1: Indefinite Integrals

Definition

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\,dx = F(x) + c, \quad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

The process of finding the indefinite integral is called integration or integrating $f(x)$. If we need to be specific about the integration variable we will say that we are integrating $f(x)$ with respect to x.

Properties of the Indefinite Integral:

$1 - \int k.f(x)dx = k\int f(x)dx,$ where k is any number. So, we can factor multiplicative constants out of indefinite integrals.

$2 - \int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx.$ In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.

!!!!! Warning !!!!!

$$\int f(x)\times g(x)\,dx \neq \int f(x)\,dx \times \int g(x)\,dx.$$

$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}.$$

The first integral that we will look at is the integral of a power of *x*.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1.$$

The general rule when integrating a power of *x* we add one onto the exponent and then divide by the new exponent. It is clear that we will need to avoid $n = -1$ in this formula. If we allow in this formula we will end up with division by zero.

Next is one of the easier integrals but always seems to cause problems for students.

$$\int k dx = kx + c, \quad c \text{ and } k \text{ are constants.}$$

Let us now take a look at the trigonometric functions:

$$\int \sin x dx = -\cos x + c.$$

$$\int \cos x dx = \sin x + c.$$

$$\int \sec^2 x dx = \tan x + c.$$

$$\int \sec x \tan x dx = \sec x + c.$$

Now, let us take care of exponential and logarithm functions.

$$\int e^x dx = e^x + c.$$

$$\int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + c.$$

Example:

$$(a) \int 5t^3 - 10t^{-6} + 4dt.$$

$$= 5\int t^3 dt - 10\int t^{-6} dt + \int 4dt =$$

$$5\frac{t^4}{4} - 10\frac{t^{-5}}{-5} + 4t + c = \frac{5t^4}{4} + 2t^{-5} + 4t + c.$$

$$(b) \int x^8 + x^{-8} dx.$$

$$= \int x^8 dx + \int x^{-8} dx = \frac{x^9}{9} + \frac{x^{-7}}{-7} + c = \frac{x^9}{9} - \frac{x^{-7}}{7} + c.$$

$$(c) \int dy.$$

$$= \int 1 dy = y + c.$$

$$(d)\int 3\sqrt[4]{x^3}+\frac{7}{x^5}+\frac{7}{6\sqrt{x}}\,dx.$$

$$=3\int\sqrt[4]{x^3}\,dx+7\int\frac{1}{x^5}\,dx+\frac{7}{6}\int\frac{1}{\sqrt{x}}\,dx$$

$$=3\int\left(x^3\right)^{\frac{1}{4}}dx+7\int x^{-5}dx+\frac{7}{6}\int x^{-\frac{1}{2}}dx$$

$$=3\,\frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1}+7\,\frac{x^{-5+1}}{-5+1}+\frac{7}{6}\,\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=\frac{12x^{\frac{7}{4}}}{7}-\frac{7x^{-4}}{4}+\frac{7x^{\frac{1}{2}}}{3}+c$$

$$(e)\int\left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw.$$

$$=\int\left(4w-w^3+4\sqrt[3]{w}-w^2\sqrt[3]{w}\right)dw.$$

$$=\int\left(4w-w^3+4w^{\frac{1}{3}}-w^{\frac{7}{3}}\right)dw.$$

$$=4\frac{w^2}{2}-\frac{w^4}{4}+4\frac{w^{\frac{4}{3}}}{\frac{4}{3}}-\frac{w^{\frac{10}{3}}}{\frac{10}{3}}+c.$$

$$=2w^2-\frac{w^4}{4}+3w^{\frac{4}{3}}-\frac{3w^{\frac{10}{3}}}{10}+c.$$

$$(f)\int\frac{4x^{10}-2x^6+15x^2}{x^3}dx.$$

$$=4\int x^7dx-2\int x^3dx+15\int x^{-1}dx$$

$$=4\frac{x^8}{8}-2\frac{x^4}{4}+15\ln|x|+c$$

$$=\frac{x^8}{2}-\frac{x^4}{2}+15\ln|x|+c$$

Integration by substitution:

After the last section we now know how to do the following integrals

$$\int \sqrt[4]{x}\,dx \qquad \int \frac{1}{t^3}\,dt \qquad \int \cos w\,dw \qquad \int e^y\,dy$$

However, we can't do the following integrals:

$$\int 18x^2 \sqrt[4]{6x^3+5}\,dx \qquad \int \frac{2t^3+1}{(t^4+2t)^3}\,dt$$

$$\int \left(1-\frac{1}{w}\right)\cos(w-\ln w)\,dw \qquad \int (8y-1)e^{4y^2-y}\,dy$$

Let us start with the first one

$$\int 18x^2 \sqrt[4]{6x^3+5}\,dx$$

In this case let us notice

$$u = 6x^3 + 5$$

and we compute the differential

$$du = u'\,dx = 18x^2\,dx$$

Now, we go back to our integral and notice that we can eliminate every x that exists in the integral and write the integral completely in terms of u using both the definition of u and its differential

$$\int 18x^2 \sqrt[4]{6x^3 + 5}\,dx = \int \left(6x^3 + 5\right)^{\frac{1}{4}} \left(18x^2\,dx\right) = \int u^{\frac{1}{4}}\,du$$

Evaluating the integral gives,

$$\int 18x^2 \sqrt[4]{6x^3 + 5}\,dx = \int u^{\frac{1}{4}}\,du = \frac{4u^{\frac{5}{4}}}{5} + c = \frac{4\left(6x^3 + 5\right)^{\frac{5}{4}}}{5} + c$$

Example:

$$(a)\int\left(1-\frac{1}{w}\right)\cos(w-\ln w)\,dw$$

$$(b)\int 3(8y-1)e^{4y^2-y}\,dy$$

$$(c)\int x^2\left(3-10x^3\right)^4 dx$$

$$(d)\int\frac{x}{\sqrt{1-4x^2}}\,dx$$

Solution:

(a) In this case let us take $u = w - \ln w$

and we compute the differential

$$du = u'\,dw = \left(1-\frac{1}{w}\right)dw$$

Now, we go back to our integral and notice that we can eliminate every w that exists in the integral and write the integral completely in terms of u using both the definition of u and its differential

$$\int\left(1-\frac{1}{w}\right)\cos(w-\ln w)\,dw = \int\cos(w-\ln w)\left(1-\frac{1}{w}\right)dw$$

$$= \int\cos(u)\,du = \sin(u)+c = \sin(w-\ln w)+c$$

$(b)\int 3(8y-1)e^{4y^2-y}dy$ In this case let us take $u = 4y^2 - y$

and we compute the differential $du = u'dy = (8y-1)dy$

Thus we get

$$\int 3(8y-1)e^{4y^2-y}dy = \int 3e^{4y^2-y}(8y-1)dy = \int 3e^u du = 3e^u + c = 3e^{4y^2-y} + c$$

$(c)\int x^2(3-10x^3)^4 dx$ In this case let us take $u = 3-10x^3$

So, $du = u'dx = -30x^2dx$

Thus we get

$$\int x^2(3-10x^3)^4 dx = \frac{1}{-30}\int (3-10x^3)^4(-30x^2)dx$$

$$= \frac{1}{-30}\int u^4 du = \frac{-1}{150}u^5 + c = \frac{-(3-10x^3)^5}{150} + c$$

$$(d)\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

In this case let us take

$$u = 1-4x^2$$

So,

$$du = u'\,dx = (-8x)dx$$

Thus,

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int (1-4x^2)^{-\frac{1}{2}}\,x\,dx = -\frac{1}{8}\int (1-4x^2)^{-\frac{1}{2}}(-8x)dx$$

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du = -\frac{1}{8}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + c$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du .$$

Example 1:

Find $\int xe^x dx$

Solution:

Let $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + \text{c}.$$

Example 2:

Find $\int x^2 e^x dx$

Solution:

Let $u = x^2$, $dv = e^x dx$. Then $du = 2x\,dx$, $v = e^x$.

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

$$= x^2 e^x - 2\left(xe^x - e^x + \text{c}\right)$$

$$= x^2 e^x - 2xe^x + 2e^x + \text{C}$$

Example 3: Find $\int e^x \cos x dx$

Solution:

Let $u = e^x$, $dv = \cos x dx$. Then $du = e^x dx$, $v = \sin x$.

Thus, $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$

Now integrate by parts again the integral $\int e^x \sin x dx.$

Let $u = e^x$, $dv = \sin x dx$. Then $du = e^x dx$, $v = -\cos x$. So,

$$\int e^x \sin x dx = e^x \cos x - \int e^x (-\cos x) dx = e^x \cos x + \int e^x \cos x dx.$$

Inserting this integral in the first one yields

$$\int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \cos x dx.$$

Bringing the last term to the left hand side and dividing by 2 gives

$$2\int e^x \cos x dx = e^x \sin x - e^x \cos x$$

$$\therefore \int e^x \cos x dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + c.$$

Example 4:

Find $\int x^4 \ln x dx.$

Solution:

Let $u = \ln x,\ dv = x^4 dx$. Then $du = \frac{1}{x}dx,\ v = \frac{x^5}{5}$.

Thus, $\int x^4 \ln x dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\frac{1}{x}dx.$

$$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4 dx$$

$$= \frac{x^5}{5}\ln x - \frac{1}{5}\frac{x^5}{5} + \text{c}$$

$$= \frac{x^5}{5}\ln x - \frac{x^5}{25} + \text{c}.$$

Example 5:

Find $\int \ln x dx$.

Solution:

Let $u = \ln x,\ dv = 1.dx$. Then $du = \frac{1}{x}dx,\ v = x.$

Thus, $\int \ln x dx = x \ln x - \int x \frac{1}{x} dx.$

$= x \ln x - \int 1.dx$

$= x \ln x - x + \text{c}$

Integration by partial fractions:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_k}{a_kx+b_k}$$

Form of the rational function	Form of the partial fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

Main Rules:

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$$

$$\int \frac{1}{(ax+b)^n}dx = \frac{1}{a(1-n)}(ax+b)^{1-n}, \quad n \neq 1$$

Distinct roots:

Example 1:

Integrate $\int \frac{x\,dx}{(x-1)(x-2)}$

Solution:

First we write

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

Now multiply this equation by $(x\text{-}1)(x\text{-}2)$, getting

$$x = A(x-2) + B(x-1)$$

If we substitute $x = 1$, we get $1 = A(1-2)$, so $A = -1$;

Now letting $x = 2$, we get $2 = B(2-1)$, so $B = 2$.

Thus

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

Integrating, we get

$$\int \frac{x\,dx}{(x-1)(x-2)} = -\int \frac{dx}{(x-1)} + 2\int \frac{dx}{(x-2)} = -\ln|x-1| + 2\ln|x-2| + c.$$

Example 2: Integrate $\int \frac{(x^2-3)dx}{(x^2-1)(x-3)}$

Solution: First we write $(x^2-1)=(x-1)(x+1)$, and

$$\frac{(x^2-3)}{(x^2-1)(x-3)}=\frac{(x^2-3)}{(x-1)(x+1)(x-3)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{D}{(x-3)}$$

Now multiply this equation by $(x-1)(x+1)(x-3)$, getting

$$(x^2-3)=A(x-1)(x-3)+B(x+1)(x-3)+D(x+1)(x-1)$$

Substitute $x=-1$, we get $1-3=A(-2)(-4)$, so $A=-1/4$;
Substitute $x=1$, we get $1-3=B(2)(-2)$, so $B=1/2$;
Substitute $x=3$, we get $9-3=C(4)(2)$, so $C=6/8=3/4$;

Therefore

$$\frac{(x^2-3)}{(x^2-1)(x-3)}=-\frac{1}{4}\frac{1}{(x+1)}+\frac{1}{2}\frac{1}{(x-1)}+\frac{3}{4}\frac{1}{(x-3)}$$

Integrating, we obtain

$$\int \frac{(x^2-3)dx}{(x^2-1)(x-3)}=-\frac{1}{4}\ln|x+1|+\frac{1}{2}\ln|x-1|+\frac{3}{4}\ln|x-3|+c.$$

Multiple roots:

Example 3: Integrate $\int \frac{(x^2+1)dx}{x^2(x-1)}$

Solution: First we write

$$\frac{(x^2+1)}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{D}{x-1}$$

Thus

$$x^2+1 = Ax(x-1) + B(x-1) + Dx^2$$

Substituting $x = 0$, we get $1 = B(-1)$, so $B = -1$.

Substituting $x = 1$, we get $2 = D$. To find A we take any other value of x;

Substitute $x = -1$, we get $2 = A(-1)(-2) + (-1)(-2) + 2$, so

$-2 = 2A$, and so $A = -1$.

Therefore

$$\frac{(x^2+1)}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

which we can integrate term by term

$$\int \frac{(x^2+1)dx}{x^2(x-1)} = -\ln|x| + \frac{1}{x} + 2\ln|x-1| + c.$$

Example 4:

$$\int \frac{(x^2+1)\,dx}{(x+3)(x-1)^2}$$

Solution:

First we write

$$\frac{(x^2+1)}{(x+3)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{x+3}$$

Multiply by $(x+3)(x-1)^2$ we get

$$x^2 + 1 = A(x+3)(x-1) + B(x+3) + D(x-1)^2$$

Substitute $x = -3$, we get $9 + 1 = D(-4)^2$, $so\, D = \frac{5}{8}$.

Substitute $x = 1$, we get $2 = B(4)$, $so\, B = \frac{1}{2}$.

Substitute $x = 0$ we get $1 = -3A + 3B + D = -3A + \frac{3}{2} + \frac{5}{8}$,

$3A = \frac{3}{2} + \frac{5}{8} - 1 = \frac{12+5-8}{8} = \frac{9}{8}$. So, $A = \frac{3}{8}$

Thus

$$\frac{(x^2+1)}{(x+3)(x-1)^2} = \frac{3}{8}\frac{1}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2} + \frac{5}{8}\frac{1}{x+3}$$

Integrating, we obtain

$$\int \frac{(x^2+1)\,dx}{(x+3)(x-1)^2} = \frac{3}{8}\ln|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\ln|x+3| + c.$$

No roots:

Main Rule:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$$

Example 5:

Evaluate $\int \frac{dx}{x^2+9}$

Solution:

$$\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+3^2} = \frac{1}{3}\tan^{-1}(\frac{x}{3}) + c$$

Example 6:

Evaluate $\int \frac{dx}{x^2-4x+5}$

Solution:

Here we can't find real factors, because the roots are complex. But we can complete the square:

$$x^2-4x+5 = (x^2-4x+4)+5-4 = (x-2)^2+1$$

Thus

$$\int \frac{dx}{x^2-4x+5} = \int \frac{dx}{(x-2)^2+1} = \tan^{-1}(x-2) + c.$$

More exercises

1) $\int (3x^2+1)\sin(x^3+x+1)\,dx$; $\int \frac{x+3}{(x-3)(x-2)}dx$; $\int \frac{5}{x^2+1}dx$;

2) $\int (x^2+1)\ln x\,dx$; $\int x^2 \sin x\,dx$; $\int \frac{x}{(x+2)^2}dx$; $\int x\sin x\,dx$

3) $\int \frac{dx}{x^2+9}$; $\int x^2\cos x\,dx$; $\int \frac{x-1}{(x-1)^2}dx$; $\int x^2 \ln x\,dx$; $\int x\ln(x)\,dx$;

4) $\int \frac{dx}{(x-3)(x-2)}$; $\int (x+1)\cos(x^2+2x)\,dx$; $\int \frac{x-2}{(x-2)(x-3)}dx$;

5) $\int \frac{x-1}{(x+2)^2(x+1)}dx$; $\int (2x+1)\sin x\,dx$; $\int \frac{\sin x}{\cos x}dx$;

6) $\int (3x^2+1)\sin(x^3+x+1)\,dx$; $\int \frac{x+3}{(x-3)(x-2)}dx$; $\int \frac{5}{x^2+1}dx$;

7) $\int (x^2+1)\ln x\,dx$; $\int \frac{2x+1}{(x+1)(x-4)}dx$; $\int \frac{2x}{x^2-1}dx$.