Chapter 6: Functions with severable variables and Partial Derivatives:

Functions of several variables: A function involving more than one variable is called function with severable variables.

Examples:

$$f(x,y)=\frac{2xy}{x^2+y^2};$$

$$f(x,y)=\frac{2}{x^2-y^2};$$

$$f(x,y)=\frac{x+y}{y-2};$$

$$f(x,y)=x^2+y^2.$$

$$f(x,y)=-x^2+y$$

$$f(x,y)=\sqrt{\sin\left(x^2+y^2\right)}$$

$$f(x,y)=\frac{1}{9}y^3\sin(x)$$

$$f(x,y)=\frac{1}{2}x^2+2y^2$$

$$f(x,y)=\sin(x)\cos(2y)$$

$$f(x,y)=\frac{1}{2}\left(x^2-y^2\right)$$

$$f(x,y)=e^{-x^2}+e^{-4y^2}$$

$$f(x,y)=\frac{-4}{1+x^2+2y^2}$$

$$f(x,y,z)=y\ln(x^2+z^2);$$

$$f(x,y,z)=z+\sqrt{x^2+y^2};$$

$$f(x,y,z)=xye^{xyz};$$

$$f(x,y,z)=yx^2+z^2+\cos(xyz).$$

Partial Derivatives: $f(x,y)$

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Example 1: $f(x,y) = 3x^2 + 2xy - y^2$ $\quad \frac{\partial f}{\partial x} = ?,\ \frac{\partial f}{\partial y} = ?$

Sol.: $\frac{\partial f}{\partial x} = 6x + 2y$ $\quad \frac{\partial f}{\partial y} = 2x - 2y$

Example 2: $f(x,y) = e^{xy}\sin(x+y)$ $\quad \frac{\partial f}{\partial x} = ?,\ \frac{\partial f}{\partial y} = ?$

Sol.:

$$\frac{\partial f}{\partial x} = ye^{xy}\sin(x+y) + e^{xy}\cos(x+y)$$

$$\frac{\partial f}{\partial y} = xe^{xy}\sin(x+y) + e^{xy}\cos(x+y)$$

Partial Derivatives $f(x,y,z)$:

$$f(x,y,z)$$

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

$$f_z = \frac{\partial f}{\partial z}$$

Example 2: $f(x,y,z) = \ln(x^2 + y^2 + z^2)$

$$\frac{\partial f}{\partial x} = ?,\ \frac{\partial f}{\partial y} = ?,\ \frac{\partial f}{\partial z} = ?$$

Sol.:

$$f_x(x,y,z) = \frac{2x}{x^2 + y^2 + z^2}$$

$$f_y(x,y,z) = \frac{2y}{x^2 + y^2 + z^2}$$

$$f_z(x,y,z) = \frac{2z}{x^2 + y^2 + z^2}$$

Higher order Partial Derivatives:

$$f(x, y)$$

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Higher order Partial Derivatives:

$$f(x, y, z)?$$

$$
f(x,y,z) \rightarrow
\begin{cases}
\dfrac{\partial f}{\partial x} = f_x \rightarrow
\begin{cases}
\dfrac{\partial}{\partial x}(f_x) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx} \rightarrow \cdots \\
\dfrac{\partial}{\partial y}(f_x) = \dfrac{\partial^2 f}{\partial y \partial x} = f_{xy} \rightarrow \cdots \\
\dfrac{\partial}{\partial z}(f_x) = \dfrac{\partial^2 f}{\partial z \partial x} = f_{xz} \rightarrow \cdots
\end{cases} \\
\dfrac{\partial f}{\partial y} = f_y \rightarrow
\begin{cases}
\dfrac{\partial}{\partial x}(f_y) = \dfrac{\partial^2 f}{\partial x \partial y} = f_{yx} \rightarrow \cdots \\
\dfrac{\partial}{\partial y}(f_y) = \dfrac{\partial^2 f}{\partial y^2} = f_{yy} \rightarrow \cdots \\
\dfrac{\partial}{\partial z}(f_y) = \dfrac{\partial^2 f}{\partial z \partial y} = f_{yz} \rightarrow \cdots
\end{cases} \\
\dfrac{\partial f}{\partial z} = f_z \rightarrow
\begin{cases}
\dfrac{\partial}{\partial x}(f_z) = \dfrac{\partial^2 f}{\partial x \partial z} = f_{zx} \rightarrow \cdots \\
\dfrac{\partial}{\partial y}(f_z) = \dfrac{\partial^2 f}{\partial y \partial z} = f_{zy} \rightarrow \cdots \\
\dfrac{\partial}{\partial z}(f_z) = \dfrac{\partial^2 f}{\partial z^2} = f_{zz} \rightarrow \cdots
\end{cases}
\end{cases}
$$

Example 3:

Find all the second partial derivatives of $f(x, y) = e^{xy}\sin(x+y)$

Sol.:

$$\frac{\partial f}{\partial x} = [y\sin(x+y) + \cos(x+y)]e^{xy}$$

$$\frac{\partial f}{\partial y} = [x\sin(x+y) + \cos(x+y)]e^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = [y\cos(x+y) - \sin(x+y)]e^{xy} + y[y\sin(x+y) + \cos(x+y)]e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = [yx\sin(x+y) + (x+y)\cos(x+y)]e^{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial y^2} = [x\cos(x+y) - \sin(x+y)]e^{xy} + x[x\sin(x+y) + \cos(x+y)]e^{xy}$$

Example 4: Find the second partial derivatives f_{xx}, f_{yy}, f_{zz} of the following function $f(x,y,z)=\ln(x^2+y^2+z^2)$

Sol.:

$$f_x(x,y,z)=\frac{2x}{x^2+y^2+z^2}$$

$$f_y(x,y,z)=\frac{2y}{x^2+y^2+z^2}$$

$$f_z(x,y,z)=\frac{2z}{x^2+y^2+z^2}$$

$$f_{xx}(x,y,z)=\frac{2(x^2+y^2+z^2)-2x(2x)}{\left(x^2+y^2+z^2\right)^2}=\frac{2y^2+2z^2-2x^2}{\left(x^2+y^2+z^2\right)^2}$$

$$f_{yy}(x,y,z)=\frac{2(x^2+y^2+z^2)-2y(2y)}{\left(x^2+y^2+z^2\right)^2}=\frac{2x^2+2z^2-2y^2}{\left(x^2+y^2+z^2\right)^2}$$

$$f_{zz}(x,y,z)=\frac{2(x^2+y^2+z^2)-2z(2z)}{\left(x^2+y^2+z^2\right)^2}=\frac{2x^2+2y^2-2z^2}{\left(x^2+y^2+z^2\right)^2}$$

Example 5: Find all the second partial derivatives of the following function

$$f(x,y) = x^3 y - 2x^2 y^2 + xy^3 + 1$$

Sol.:

$$f_x(x,y) = 3x^2 y - 4xy^2 + y^3$$

$$f_y(x,y) = x^3 - 4x^2 y + 3xy^2$$

$$f_{xx}(x,y) = 6xy - 4y^2$$

$$f_{yy}(x,y) = -4x^2 + 6xy$$

$$f_{xy}(x,y) = 3x^2 - 8xy + 3y^2 = f_{yx}(x,y)$$

Example 6: Find the partial derivatives w_{yx}, w_{xy}, w_y, w_x at $(2,0)$ of the function

$$w = x^2 y e^{-xy}.$$

Sol.:

$$w_x = 2xye^{-xy} - x^2y^2e^{-xy} \qquad w_y = x^2e^{-xy} - x^3ye^{-xy}.$$

$$w_{xy} = [2xe^{-xy} + 2xye^{-xy}(-x)] - [x^2 2ye^{-xy} + x^2y^2e^{-xy}(-x)].$$

$$= 2xe^{-xy}(1-xy) - x^2ye^{-xy}(2-xy).$$

$$= xe^{-xy}[2(1-xy) - xy(2-xy)]$$

$$= xe^{-xy}[2 - 2xy - 2xy + x^2y^2] = xe^{-xy}[2 - 4xy + x^2y^2]$$

$$w_{yx} = [2xe^{-xy} + x^2e^{-xy}(-y)] - [3x^2ye^{-xy} + x^3ye^{-xy}(-y)].$$

$$= xe^{-xy}(2-xy) - x^2ye^{-xy}(3-xy).$$

$$= xe^{-xy}[(2-xy) - xy(3-xy)] = xe^{-xy}[2 - 4xy + x^2y^2]$$

$$\therefore w_x\Big|_{(2,0)} = 0 \quad w_y\Big|_{(2,0)} = 4 - 0 = 4 \quad w_{xy}\Big|_{(2,0)} = 2(2) = 4 = w_{yx}\Big|_{(2,0)}$$

Chain Rule:

Given a function $w=f(x,y,z)$, $x=g(s,t)$, $y=h(s,t)$, $z=k(s,t)$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{ds}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{ds}$$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{dt}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{dt}$$

Example 1:

$w=x+2y+z^2,\quad x=\frac{s}{t},\ y=s^2+\ln t,\ z=2s$ Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$?

Sol.:

$$\frac{\partial w}{\partial x}=1,\ \frac{\partial w}{\partial y}=2,\ \frac{\partial w}{\partial z}=2z,\ \frac{\partial x}{\partial s}=\frac{1}{t},\ \frac{\partial y}{\partial s}=2s,\ \frac{\partial z}{\partial s}=2,\ \frac{\partial x}{\partial t}=-\frac{s}{t^2},\ \frac{\partial y}{\partial t}=\frac{1}{t},\ \frac{\partial z}{\partial t}=0$$

$$\therefore\ \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{ds}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{ds}=(1)(\frac{1}{t})+(2)(2s)+(2z)(2)=\frac{1}{t}+4s+4z$$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{dt}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{dt}=(1)(-\frac{s}{t^2})+(2)(\frac{1}{t})+(2z)(0)=\frac{-s+2t}{t^2}$$

Example 2: Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ in the following cases:

1) $w = x\sin y, \quad x = s^2 + t^2, \; y = st.$

2) $w = x^2 + 2xy, \quad x = s\ln t, \; y = 2s + t.$

3) $w = x\ln y, \quad x = 3s + t, \; y = st.$

4) $w = x^2\cos y, \quad x = s^2 t, \; y = s - 1.$

5) $w = xy + yz, \quad x = 2s - t, \; y = s - 2t, z = -2s + 2t.$

Solution:

1) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s}$ $\quad \frac{\partial w}{\partial x} = \sin y, \frac{\partial x}{\partial s} = 2s, \frac{\partial w}{\partial y} = x\cos y, \frac{\partial y}{\partial s} = t.$

Thus, $\frac{\partial w}{\partial s} = \sin y.(2s) + x\cos y.(t) = 2s\sin(st) + t(s^2 + t^2)\cos(st).$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial t}$ $\quad \frac{\partial x}{\partial t} = 2t, \frac{\partial y}{\partial t} = s.$

Thus, $\frac{\partial w}{\partial t} = \sin y.(2t) + x\cos y.(s) = 2t\sin(st) + s(s^2 + t^2)\cos(st).$

2) $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s}$$ $$\frac{\partial w}{\partial x} = 2x + 2y, \frac{\partial x}{\partial s} = \ln t, \frac{\partial w}{\partial y} = 2x, \frac{\partial y}{\partial s} = 2.$$

$$\text{Thus, } \frac{\partial w}{\partial s} = (2x+2y).(\ln t) + 2x.(2)$$
$$= (2s\ln t + 4s + 2t)\ln t + 4s\ln t$$
$$= 2[s\ln t + 4s + t]\ln t.$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial t}$$ $$\frac{\partial x}{\partial t} = \frac{s}{t}, \frac{\partial y}{\partial t} = 1.$$

$$\text{Thus, } \frac{\partial w}{\partial t} = (2x+2y).(\frac{s}{t}) + 2x.(1)$$
$$= (2s\ln t + 4s + 2t)\frac{s}{t} + 2s\ln t$$

3,4,5) *Quizes*.

Differentiation of Implicit functions:

Assume that z is an implicit function of x and y, that is, $F(x,y,z)=0$. Then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}.$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}.$$

Example 1: If $z^3 - xy + yz + y^3 - 2 = 0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

Solution:

Let $F(x,y,z) = z^3 - xy + yz + y^3 - 2$. Then

$$F_x = -y,\ F_y = -x + z + 3y^2, F_z = 3z^2 + y.$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{-y}{3z^2 + y} = \frac{y}{3z^2 + y},\ \frac{\partial z}{\partial y} = -\frac{-x + z + 3y^2}{3z^2 + y} = \frac{x - z - 3y^2}{3z^2 + y}.$$

Example 2:

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in the following cases:

1) $x\sin y + z^2 = 2xyz.$

2) $2xz^3 - 3yz^2 + x^2y^2 + 4z = 0.$

3) $xz^2 + 2x^2y - 4y^2z + 3y = 2.$

4) $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1.$

5) $yx^2 + z^2 + \cos(xyz) = 4.$

Solution:

1) Let $F(x,y,z) = x\sin y + z^2 - 2xyz.$ Then

$$F_x = \sin y - 2yz,\ F_y = x\cos y - 2xz, F_z = 2z - 2xy.$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sin y - 2yz}{2z - 2xy}\ ,\ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x\cos y - 2xz}{2z - 2xy}.$$

2,3,4,5) *Quizes.*

Chapter 7: Introductory to Differential Equations:

Definition: A differential equation (D.E.) is an equation involving variables and derivatives.

Examples:

$$y' = 3x^2$$

$$(y''')^4 - x^2 y = e^x$$

$$1 - y^{(4)} = x^5 y^3$$

The variable y is called the solution of the differential equation.

For example the function $y = x^3 + C$ is a solution of the first one.

Separable Differential Equations:

Definition: A differential equation (D.E.) is called separable if it can be written in the form: $M(x)dx+N(y)dy=0$.

Example 1: Solve the differential equation $3y^2e^{3x}+\dfrac{dy}{dx}=0$

Sol.:

$$3y^2e^{3x}+\frac{dy}{dx}=0$$

$$3y^2e^{3x}dx+dy=0$$

$$3e^{3x}dx+\frac{1}{y^2}dy=0, \quad (y\neq 0)$$

$$e^{3x}-\frac{1}{y}=C$$

$$y=\frac{1}{e^{3x}-C}$$

Example 2: Solve the differential equation $x^2dy + y^2dx = 0$

Sol.:

$x^2dy + y^2dx = 0,$ Divide by x^2 and y^2, $x \neq 0, y \neq 0.$

$$\frac{1}{y^2}dy + \frac{1}{x^2}dx = 0$$

$$y^{-2}dy + x^{-2}dx = 0$$

$$-y^{-1} + x^{-1} = C$$

$$\frac{1}{y} = \frac{1}{x} - C \Rightarrow y = \frac{1}{\frac{1}{x} - C} = y = \frac{x}{1 - Cx}$$

Remark: This solution is called the general solution of the differential equation. If we add to the previous example the boundary condition $y(2)=-2$. Then substitute the values of x and y into the general solution

$$y = \frac{x}{1 - Cx}$$

we get $C=1$ and so the obtained solution is called the particular solution.

$$y = \frac{x}{1 - x}$$

Linear First order Differential Equations:

Definition: A differential equation (D.E.) is called Linear First order Differential Equations if it can be written in the form:

$$y' + P(x)y = Q(x)$$

Example 1: The following differential equations are L.F.O.D.E.:

(a) $y' + y\cos x = \cos x \sin x.$

(b) $y' - \dfrac{ny}{x+1} = e^{x}(x+1)^{n}.$

(c) $x(x-1)y' + (1-2x)y + x^2 = 0.$

(d) $y' - \dfrac{2}{x}y = x^4.$

(e) $(1+x^2)y' + xy = \dfrac{1}{1+x^2}.$

Solving a First Order Linear Differential Equation

1) Put the equation into standard form : $\frac{dy}{dx} + P(x)y = Q(x)$.

2) Identify $P(x)$ and $Q(x)$.

3) Find $\int P(x)dx$.

4) Let $\rho(x) = e^{\int P(x)dx}$.

$$y = \frac{\int Q(x)\rho(x)dx + C}{\rho(x)}$$

Example 2: Solve the differential equation: $xy'+2y-2x^3=0$

Solution:

$xy'+2y-2x^3=0$, divide by $x\neq 0$

$$y'+\frac{2}{x}y=2x^2,$$

$$y'+P(x)y=Q(x).$$

with $P(x)=\frac{2}{x},\ Q(x)=2x^2,$

$$\rho(x)=e^{\int P(x)dx}=e^{2\int\frac{1}{x}dx}=e^{2\ln|x|}=x^2.$$

$$e^{n\ln|x|}=x^n.$$

$$\int\rho(x)Q(x)dx=\int x^2(2x^2)dx=\int x^4dx=\frac{x^5}{5}+C.$$

Thus

$$y=\frac{1}{\rho(x)}\left[\int\rho(x)Q(x)dx+C\right]=\frac{1}{x^2}\left(\frac{2x^5}{5}+C\right)=\frac{2x^3}{5}+\frac{C}{x^2}.$$

Example 3: Solve the following differential equations:

1) $y'+3y=e^{-2x}.$

2) $xy'+y=\sin x, \quad y(\frac{\pi}{2})=1.$

3) $xy'-3y=x^2 \quad (x>0).$

4) $\frac{dy}{dx}-xy=x, \quad y(0)=3.$

5) $\frac{dy}{dx}+y-\frac{1}{1+e^x}=0.$

6) $y'+2y=x, \quad y(0)=1.$

7) $y'+y=\cos(e^x).$

8) $x\frac{dy}{dx}+2y=x^3, \quad y(2)=1.$

9) $2\frac{dy}{dx}+10y=1.$

Solution: 1) $y'+P(x)y=Q(x).$ with $P(x)=3,\ Q(x)=e^{-2x},$

$$\rho(x)=e^{\int 3dx}=e^{3x}.$$

$$\int \rho(x)Q(x)dx=\int e^{3x}(e^{-2x})dx=\int e^{x}dx=e^{x}+C.$$

$$y=\frac{1}{\rho(x)}\left[\int \rho(x)Q(x)dx+C\right]=\frac{1}{e^{3x}}\left(e^{x}+C\right)=e^{-2x}+Ce^{-3x}.$$

2,3,4,5,6,7,8,9) *Quizes*.