Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Chapter 2: Multiple Integrals

- 2.1 Double Integrals

Review of the definite integral:

First let us recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ *is* defined for $a \leq x \leq b$, we start by dividing the interval $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$ and we choose sample points x_i* in these subintervals. Then we form the Riemann sum

$$\sum_{i=1}^{\infty} f(x_i^*)\Delta x$$

and take the limit of such sums as to obtain the definite integral of f from a to b:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

• 2.1 Double Integrals

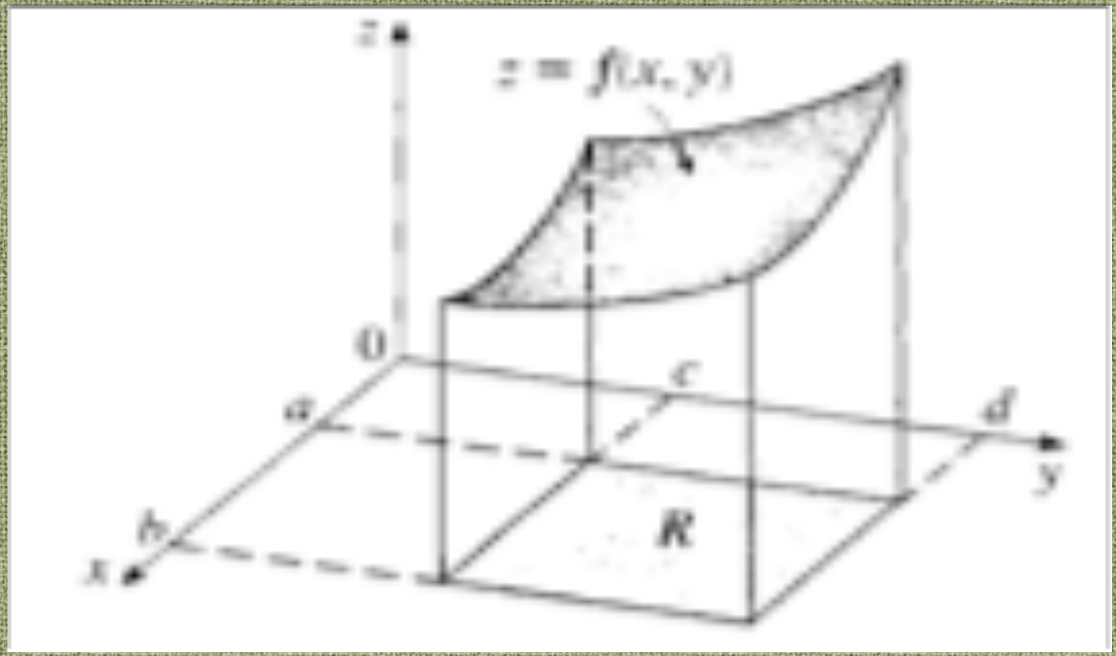
In a similar way we consider a function of two variables defined on a closed recta ngle

$$R = [a,b]\times[c,d] = \{(x,y) : a \le x \le b, c \le y \le d\}$$

and we first suppose that $f(x,y) \ge 0$. The graph of f is a surface with equation $z=f(x,y)$.

Let S be thye solid that lies above R and under the graph of f, that is,

$$S = \{(x,y,z) : 0 \le z \le f(x,y),\ (x,y) \in R\}$$



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Department of Mathematics
King Saud University

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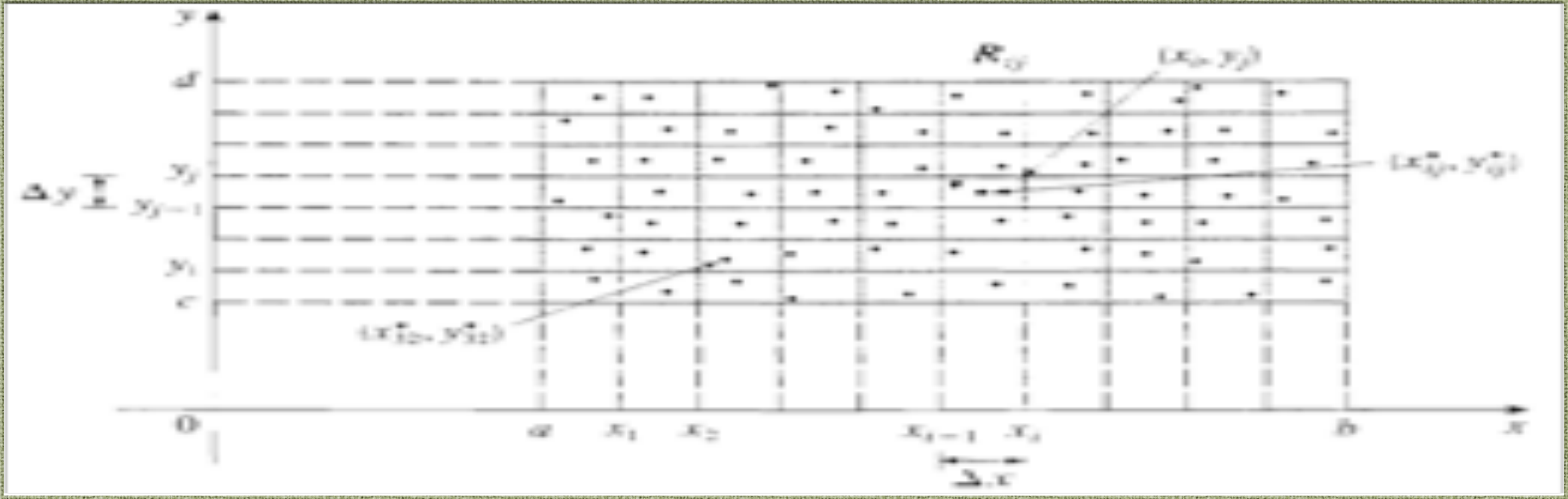
Department of Mathematics
King Saud University

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a,b]$ into m subintervals $[x_{i-1},x_i]$ of equal width $\Delta x=(b-a)/m$ and by dividing the interval $[c,d]$ into n subintervals $[y_{i-1},y_i]$ of equal width $\Delta y=(d-c)/n$



$$R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]=\{(x,y):x_{i-1}\le x\le x_i,y_{i-1}\le y\le y_i\}\ \textit{with area}\ \Delta A=\Delta x\Delta y.$$

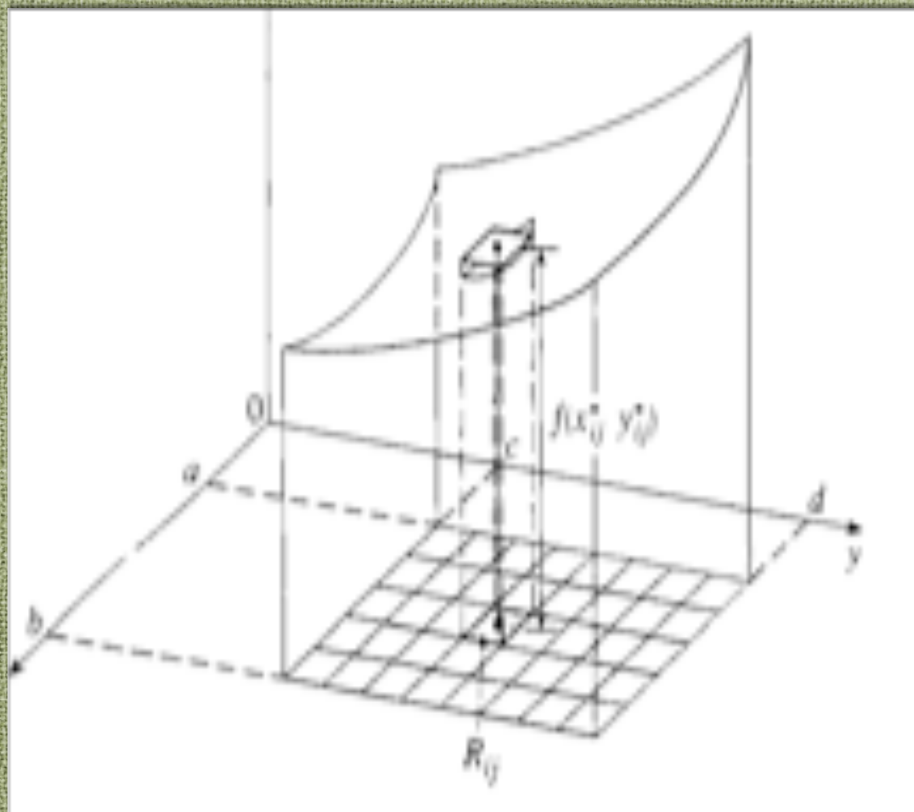
If we choose a **sample point** $(x_{i,j}^*,y_{i,j}^*)$ in each R_{ij}, thenwe can approximate the part of S that lies above each R_{ij} by a thin rectangular box with base R_{ij} and height $f(x_{i,j}^*,y_{i,j}^*)$ as shown in the opposite figure. The volume of this box is the height of the box times the area of the base rectangle: $f(x_{i,j}^*,y_{i,j}^*)\Delta A$

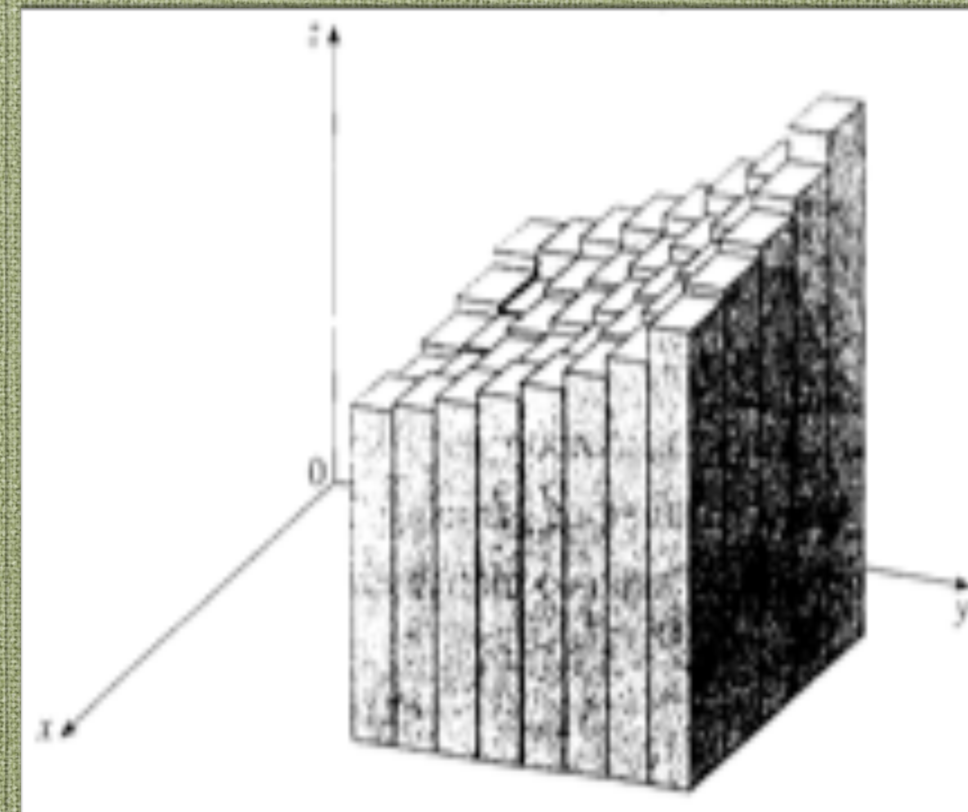
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

If we follow this procedure for all the rectangle and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}^{*}, y_{i,j}^{*}) \Delta A$$





Definition : The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i,j}^{*}, y_{i,j}^{*}) \Delta A$$

if this limit exists.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Properties of double integrals

$$\iint_R f(x,y)+g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA =$$

$$\iint_R c.f(x,y)\,dA = c.\iint_R f(x,y)\,dA,$$ where c *is a constant*

If $f(x,y) \geq g(x,y)$ *for all* $(x,y) \in R$, *then*

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA,$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Fubini's Theorem: If f is continuous on the rectangle ,

$$R=\{(x,y): a\le x\le b, c\le y\le d\}$$

then

$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

Example: Evaluate the double integral $\iint_R (x-3y^2)\,dA$, where

$$R=\{(x,y): 0\le x\le 2,\ \ 1\le y\le 2\}$$

Solution1: Fubini's Theorem gives

$$\iint_R (x-3y^2)\,dA=\int_0^2\int_1^2 (x-3y^2)\,dy\,dx=\int_0^2\left[xy-y^3\right]_{y=1}^{y=1}dx$$

$$=\int_0^2 (x-7)\,dx=\left[\frac{x^2}{2}-7x\right]_{x=0}^{x=2}=-12$$

Solution2: Again applying Fubini's Theorem, but this time integrating $w.r.t.\ x$ first we have

$$\iint_R (x-3y^2)\,dA=\int_1^2\int_0^2 (x-3y^2)\,dx\,dy=\int_1^2\left[\frac{x^2}{2}-3xy^2\right]_{x=0}^{x=1}dy$$

$$=\int_1^2 \left(2-6y^2\right)dy=\left[2y-2y^3\right]_{y=1}^{y=2}=-12$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3: Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1,2]\times\left[0,\frac{\pi}{2}\right]$

Solution 1: If we first integrate with respect to x, we get

$$\iint_R y\sin(xy)\,dA = \int_0^{\frac{\pi}{2}}\int_1^2 y\sin(xy)\,dx dy = \int_0^{\frac{\pi}{2}}[-\cos(xy)]_{x=1}^{x=2} dy$$

$$= \int_0^{\frac{\pi}{2}}(-\cos(2y)+\cos y)dy = \left[-\frac{1}{2}\sin 2y + \sin y\right]_{y=0}^{y=\frac{\pi}{2}}$$

$$= \left(-\frac{1}{2}\sin\pi + \sin\frac{\pi}{2}\right) - \left(-\frac{1}{2}\sin 0 + \sin 0\right) = 1.$$

Solution 2: If we reverse the order of integration, we get

$$\iint_R y\sin(xy)\,dA = \int_1^2\int_0^{\frac{\pi}{2}} y\sin(xy)\,dy dx\,.$$

To evaluate the inner integral $\int_0^{\frac{\pi}{2}} y\sin(xy)\,dy$, we use integration by parts with

$$u = y, \qquad dv = \sin(xy)\,dy \Longrightarrow \quad du = dy, \qquad v = -\frac{\cos(xy)}{x}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

and so

$$= \int_0^{\frac{\pi}{2}} y\sin(xy)\,dy = \left[-\frac{y\cos(xy)}{x}\right]_{y=0}^{y=\frac{\pi}{2}} + \frac{1}{x}\int_0^{\frac{\pi}{2}} \cos(xy)\,dy$$

$$= \left(-\frac{\frac{\pi}{2}\cos\left(\frac{x\pi}{2}\right)}{x}\right) + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\frac{\pi}{2}}$$

$$= \left(-\frac{\pi}{2x}\cos\left(\frac{x\pi}{2}\right)\right) + \frac{1}{x^2}\sin\left(\frac{x\pi}{2}\right)$$

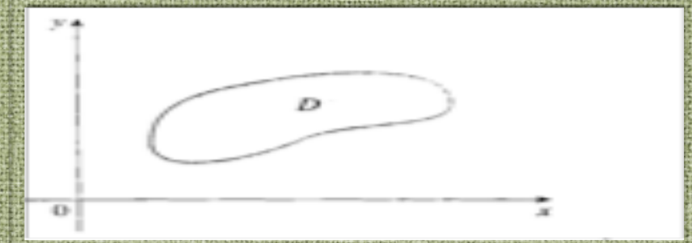
and so

$$\int_1^2 \int_0^{\frac{\pi}{2}} y\sin(xy)\,dydx = \int_1^2 \left(-\frac{\pi}{2x}\cos\left(\frac{x\pi}{2}\right)\right) + \frac{1}{x^2}\sin\left(\frac{x\pi}{2}\right) dx$$

$$= \left[-\frac{1}{x}\sin\left(\frac{x\pi}{2}\right)\right]_{x=1}^{x=2} = \left[-\frac{1}{2}\sin(\pi) + \sin\left(\frac{\pi}{2}\right)\right] = 1.$$

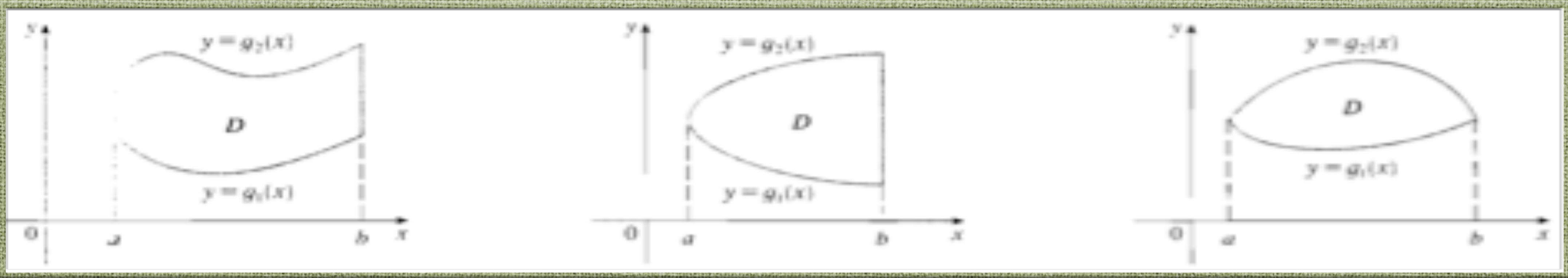
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Double integrals over general regions



A plane D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y): a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$



If f is continuous on a **type I** region D such that,

$$D = \{(x,y): a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\};$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx .$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

We also consider plane regions of **type II,** which can be expressed as

$$D=\{(x,y): c\le y\le d, h_1(y)\le x\le h_2(y)\};$$

where h_1 and h_2 are continuous.

If f is continuous on a **type II** region D, then

$$\iint\limits_D f(x,y)dA=\int_c^d\int_{h_1(x)}^{h_2(x)} f(x,y)dxdy\,.$$

Example 1: Evaluate $\iint_R (x+2y)\,dA$, where $D=[1,2]\times\left[0,\frac{\pi}{2}\right]$, is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$.

Solution: The parabolas intersect when $2x^2=1+x^2$, that is, $x^2=1$, so $x=\pm1$.

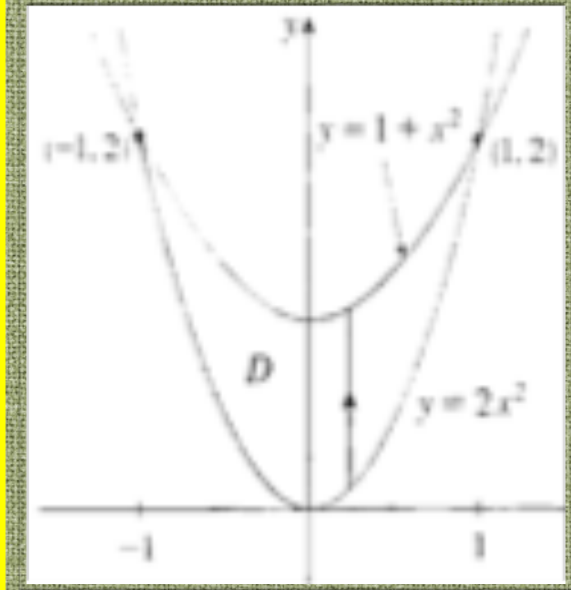
$$D=\{(x,y): -1\le x\le 1, 2x^2\le y\le 1+x^2\};$$

$$\iint\limits_D (x+2y)dA=\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx=\int_{-1}^{1}[xy+y^2]_{2x^2}^{1+x^2}dx$$

$$=\int_{-1}^{1}(x(1+x^2)+(1+x^2)^2)-(x(2x^2)+(2x^2)^2)dx$$

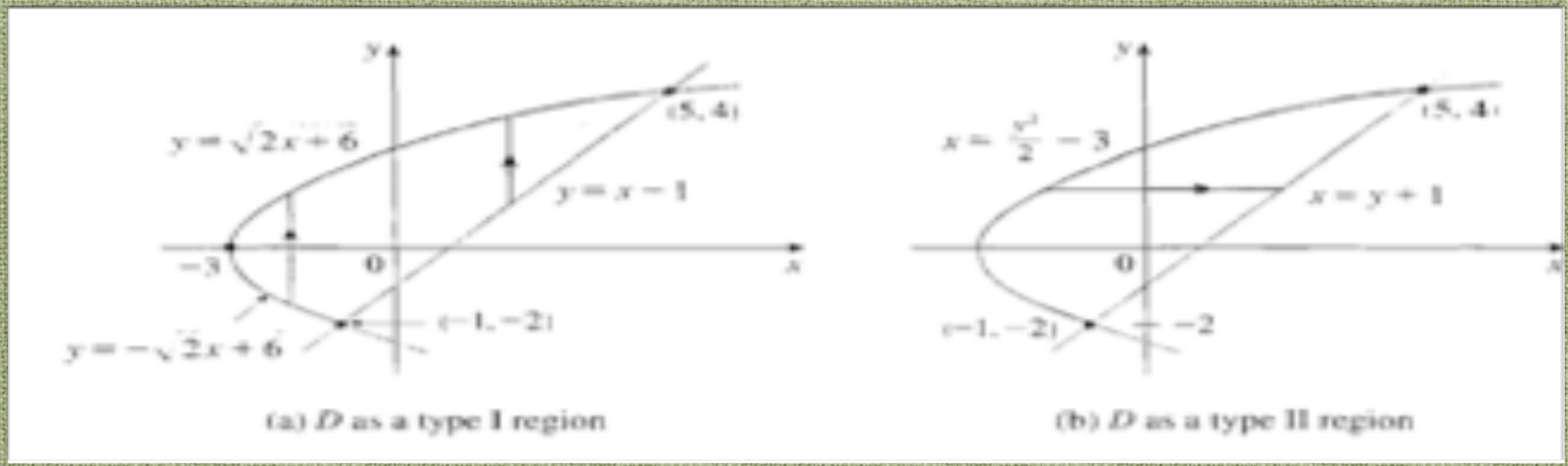
$$=\int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)dx$$

$$=\left[-\frac{3}{5}x^5-\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{1}=\frac{32}{15}.$$



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2: Evaluate $\iint_R xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



(a) D as a type I region (b) D as a type II region

$$D = \left\{(x, y) \colon -2 \le y \le 4, \quad \frac{1}{2}y^2 - 3 \le x \le y + 1\right\};$$

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4} [yx^2]_{\frac{1}{2}y^2-3}^{y+1}\,dy$$

$$= \int_{-2}^{4}\left(y(y+1)^2 - y\left(\frac{y^2}{2} - 3\right)^2\right)dy$$

$$= \int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right)dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + \frac{2y^3}{3} - 4y^2\right]_{-2}^{4} = 36.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy dx$.

Solution: $\int_0^1 \int_x^1 \sin(y^2)\, dy dx = \iint\limits_D \sin(y^2)\, dA$, *with*

$D = \{(x, y) : 0 \le x \le 1, \quad x \le y \le 1\}$, so

$D = \{(x, y) : 0 \le y \le 1, \quad 0 \le x \le y\}$;

$$\int_0^1 \int_x^1 \sin(y^2)\, dy dx = \iint\limits_D \sin(y^2)\, dA = \int_0^1 \int_0^y \sin(y^2)\, dx dy$$

$$= \int_0^1 \int_0^y \sin(y^2)\, dx dy = \int_0^1 [x \sin(y^2)\,]_{x=0}^{x=y}\, dy = \int_0^1 y \sin(y^2)\, dy$$

$$= -\frac{1}{2}[\cos(y^2)\,]_{y=0}^{y=1} = \frac{1-\cos(1)}{2}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Properties of double integrals

$$\iint_D f(x,y)+g(x,y)\,dA=\iint_D f(x,y)\,dA+\iint_D g(x,y)\,dA$$

$$\iint_D c.f(x,y)\,dA=c.\iint_D f(x,y)\,dA,\quad \text{where } c \text{ is a constant}$$

$$\text{If}\quad f(x,y)\geq g(x,y)\ \text{ for all } (x,y)\in D,$$

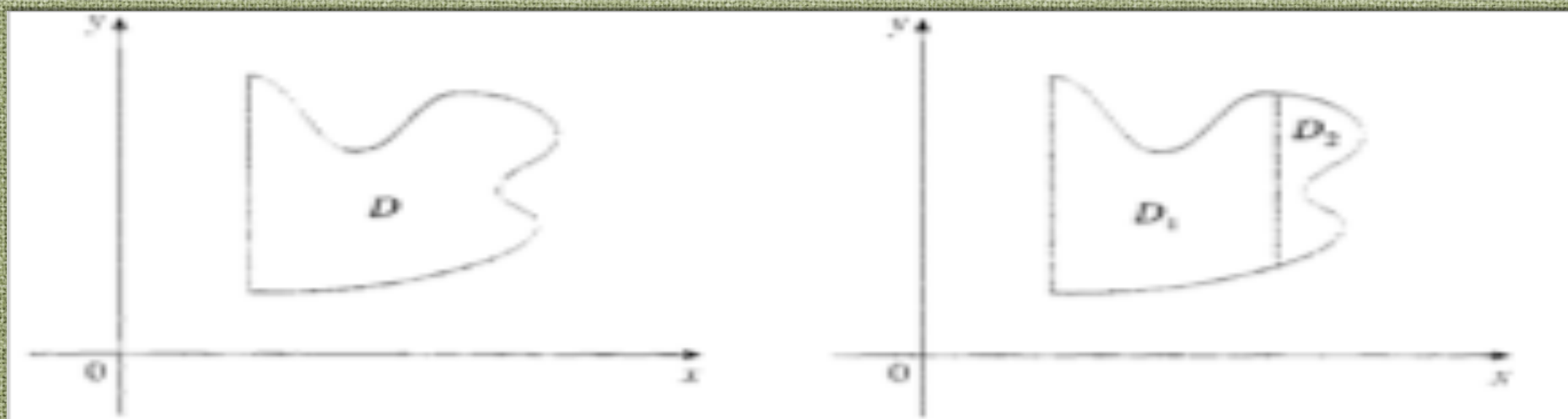
$$\text{then}\quad \iint_D f(x,y)\,dA\geq \iint_D g(x,y)\,dA.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$\iint_D f(x,y) + g(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA,$$

where $D = D_1 \cup D_2$,

and D_1 and D_2 don't overlape except perheps on their boundaries



(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of double integrals says that if we integrate the constant function $f(x,y)=1$ over a region D, we get the area of D:

$$\iint_D 1.dA = Area(D)$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Our first application is to the volume between two surfaces

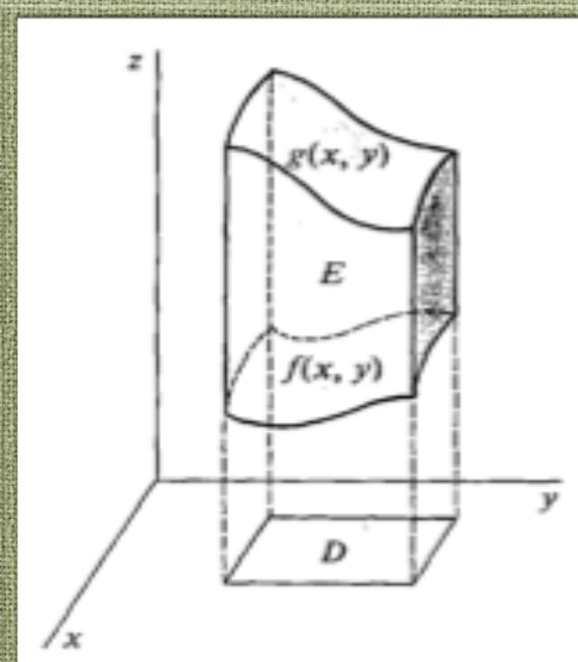
Let $f(x,y) \leq g(x,y)$ for (x,y) in D and let E be the set of all points in space such that

$$(x,y) \in D: \quad f(x,y) \leq z \leq g(x,y).$$

The ***volume*** of E is

$$V = \iint_D g(x,y) - f(x,y)\,dA\,.$$

V is called the volume over D between the surfaces $z = f(x,y)$ and $z = g(x,y)$.



Volume between two surfaces

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1: Find the volume of the solid that lies under the parabolid valuate $z = x^2 + y^2$ and above the region D in the $xy-$plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

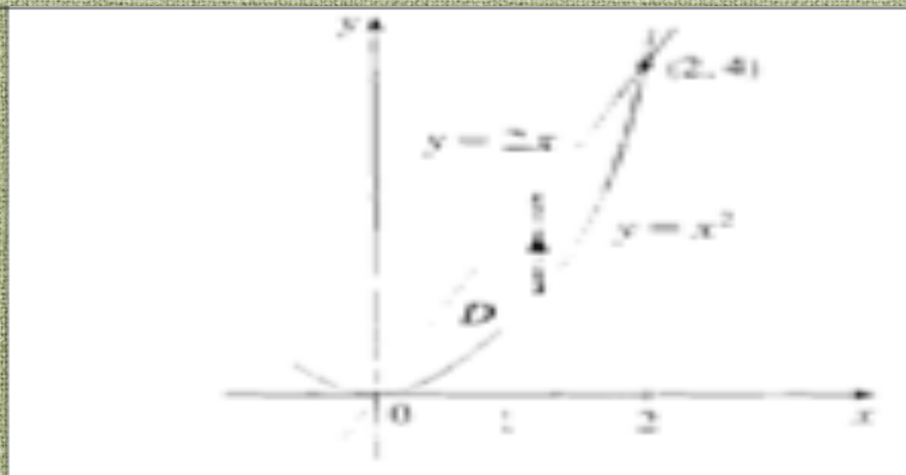


FIGURE 9
D as a type I region

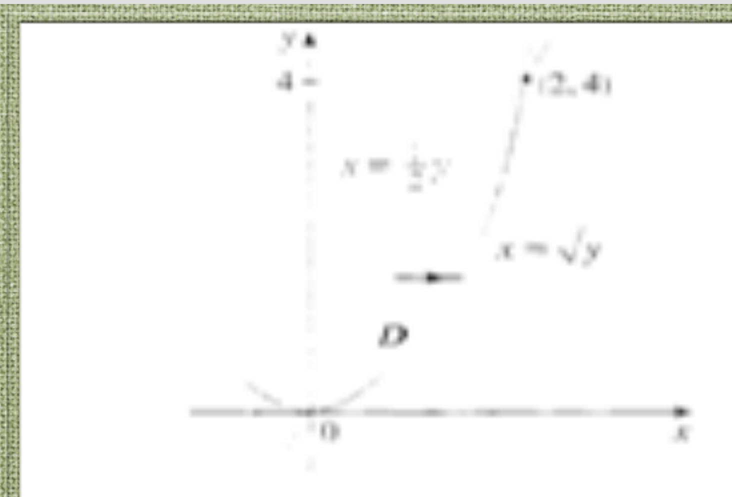


FIGURE 10
D as a type II region

$$D = \{(x,y): 0 \le x \le 2, x^2 \le y \le 2x\}$$

$$D = \{(x,y): 0 \le y \le 4, \frac{y}{2} \le x \le \sqrt{y}\}$$

$$V = \iint_D (x^2+y^2)\,dA = \int_0^2\int_{x^2}^{2x}(x^2+y^2)\,dy\,dx = \int_0^2\left[x^2y+\frac{y^3}{3}\right]_{y=x^2}^{y=2x}dx =$$

$$\int_0^2\left[\left(x^2(2x)+\frac{(2x)^3}{3}\right)-\left(x^2x^2+\frac{(x^2)^3}{3}\right)\right]dx = \int_0^2\left(-\frac{x^6}{3}-x^4+\frac{14x^3}{3}\right)dx$$

$$= \left[-\frac{x^7}{21}-\frac{x^5}{5}+\frac{7x^4}{6}\right]_0^2 = \frac{216}{35}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$D=\{(x,y):0\le y\le 4,\frac{y}{2}\le x\le\sqrt{y}\}$$

$$V=\iint_D (x^2+y^2)\,dA=\int_0^4\int_{\frac{y}{2}}^{\sqrt{y}}(x^2+y^2)\,dx\,dy=\int_0^4\left[\frac{x^3}{3}+y^2x\right]_{x=\frac{y}{2}}^{x=\sqrt{y}}dy=$$

$$\int_0^4\left[\left(\frac{(\sqrt{y})^3}{3}+y^2\sqrt{y}\right)-\left(\frac{(\frac{y}{2})^3}{3}+y^2\frac{y}{2}\right)\right]dy=\int_0^4\left(\frac{y^{\frac{3}{2}}}{3}+y^{\frac{5}{2}}-\frac{y^3}{24}-\frac{y^3}{2}\right)dy$$

$$=\left[\frac{2y^{\frac{5}{2}}}{15}+\frac{2y^{\frac{7}{2}}}{7}-\frac{13y^4}{96}\right]_0^4=\frac{216}{35}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Double integrals in polar coordinates

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

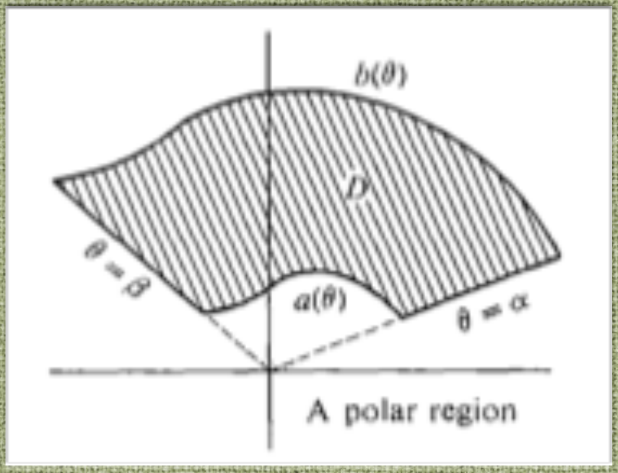
A point with polar coordinate (θ, r) has rectangular coordinates:
$(x, y) = (r\cos\theta\,, r\sin\theta\,)$.

A ***polar region*** is a region D in the $xy-$plane given by polar coordinate inequalities:

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta),$$

Where $a(\theta)$ and $b(\theta)$ are continuous. To avoid overlaps, we also require that for all $(\theta, r) \in D$,

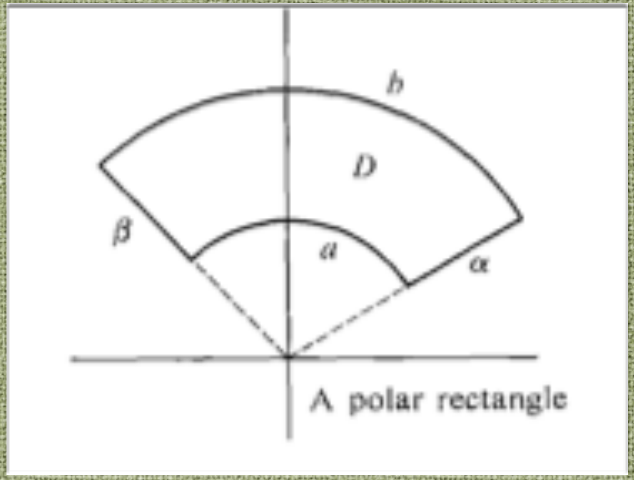
$$0 \le \theta \le 2\pi \text{ and } 0 \le r.$$



A polar region

The simplest polar regions are the polar rectangles:

$$\alpha \le \theta \le \beta, \qquad a \le r \le b.$$



A polar rectangle

Polar integration formula

Let D be the polar region $\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta).$

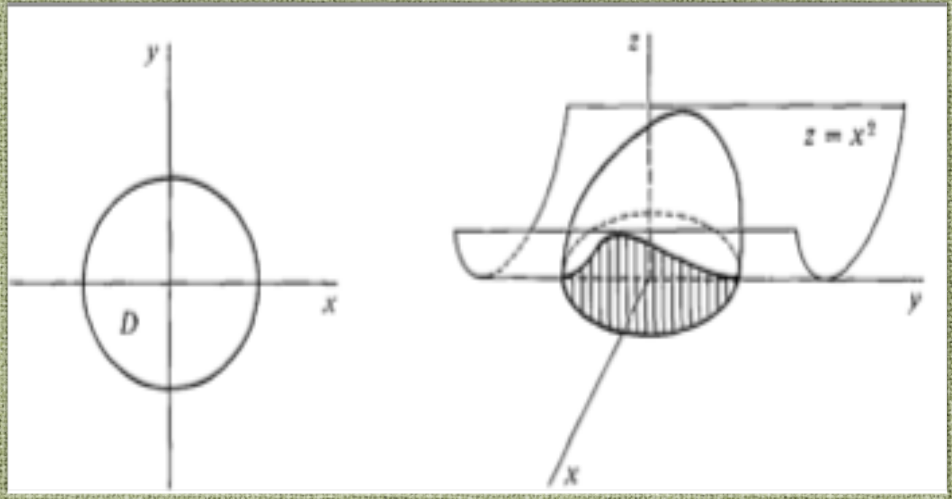
The double integral of $f(x, y)$ over D is

$$\iint_D f(x, y)\,dA = \int_{\alpha}^{\beta}\int_{a(\theta)}^{b(\theta)} f(x, y)\,r\,dr\,d\theta = \int_{\alpha}^{\beta}\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1: Find the volume over the unit circle between the surfaces $z = 0$ and $z = x^2$.



Solution.

Step 1: Sketch D and the solid,

Step 2: D is the polar region $0 \le \theta \le 2\pi, \; 0 \le r \le 1$.

Step 3:

$$V = \iint\limits_D x^2 dA = \int_0^{2\pi} \int_0^1 (r\cos\theta)^2 r dr d\theta = \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \, dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} \cos^2\theta \right]_0^1 d\theta = \frac{1}{4} \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{4} \int_0^{2\pi} \frac{[1 + \cos(2\theta)]}{2} d\theta$$

$$= \frac{1}{8} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{2\pi} = \frac{1}{8} [(2\pi) - (0)]_0^{2\pi} = \frac{\pi}{4}.$$

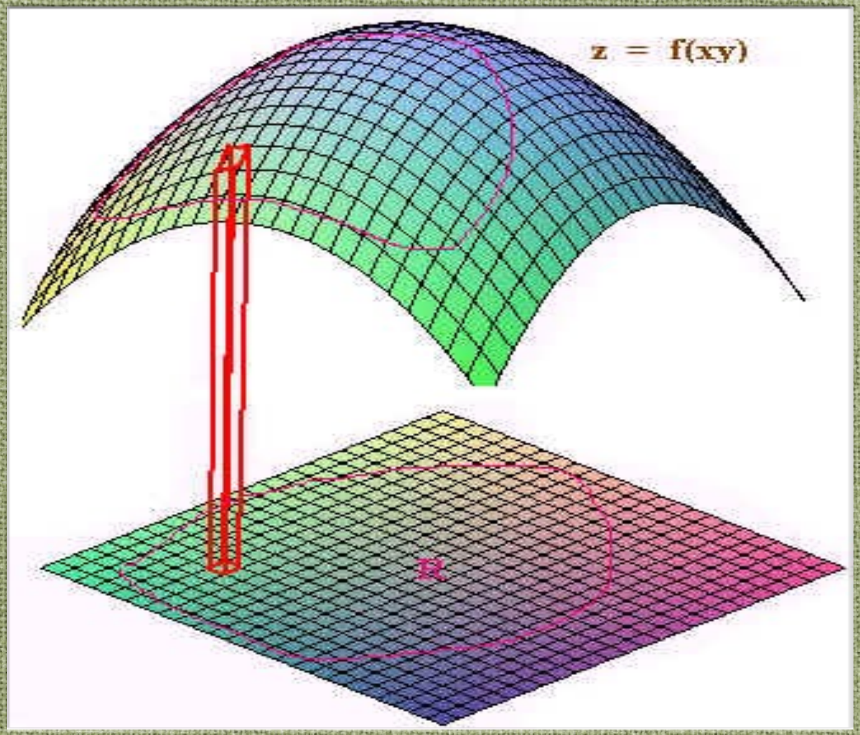
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$Surface\ Area$:

Here we are going to find the surface area of the surface gven by $z = f(x,y)$ where (x,y) is a point from the region D in the xy -plane. In this case the surface is given by :

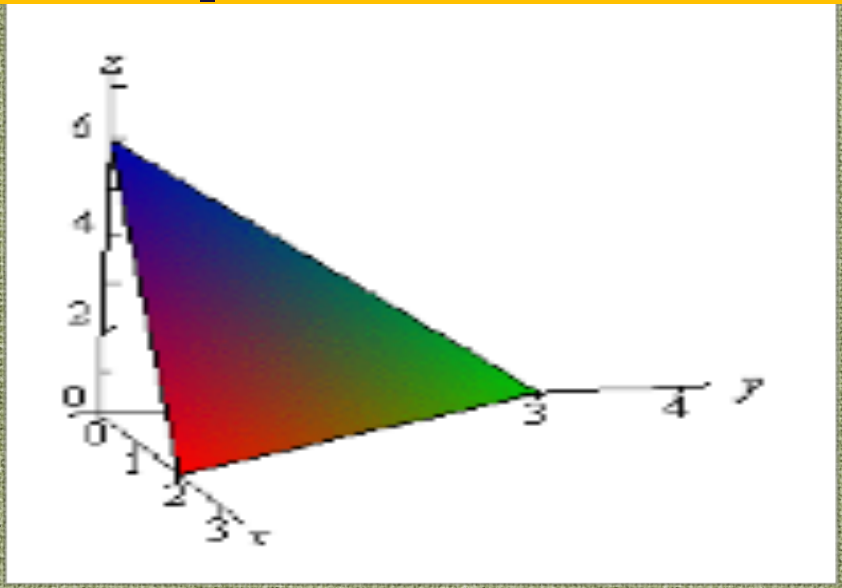
$$S = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1}\; dA$$

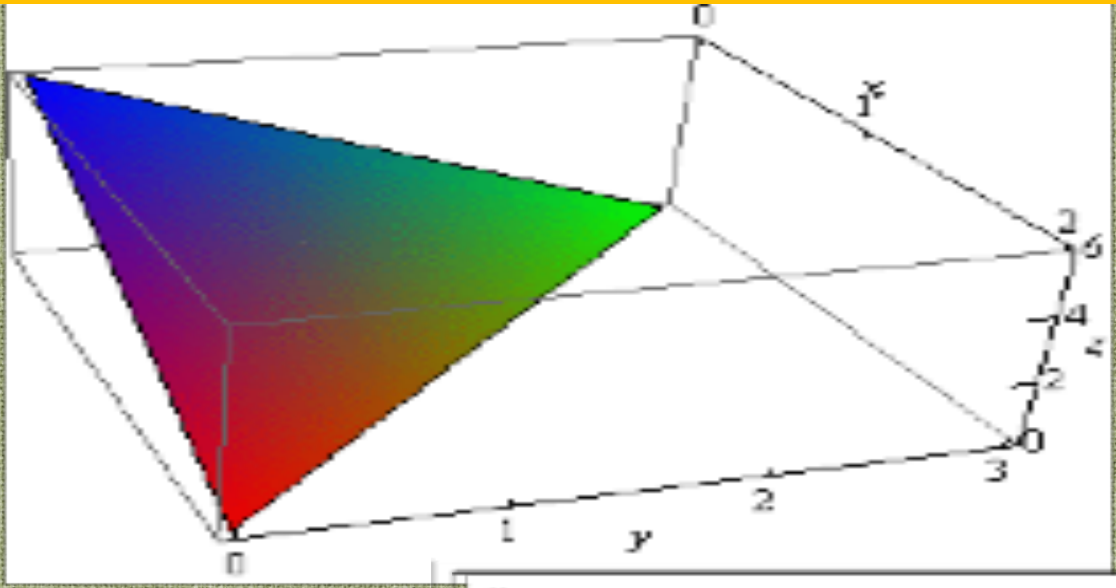


Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

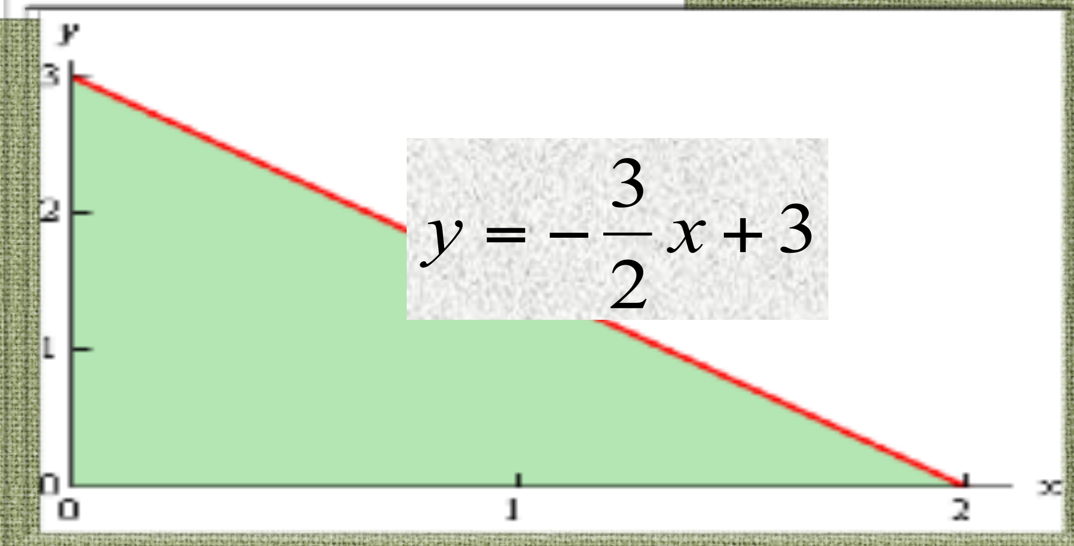
Example 1 Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Solution. Remember that the first octant is the portion of the xyz system in which all three variables are positive. Let us first get a sketch of the part of the plane that we are interested in.





We will also need a sketch of the region D.



Notice that in order to use the surface area formula we need to have the function in the form $z = f(x, y)$ and so solving for and taking the partial derivatives gives.

$$z = 6 - 3x - 2y, \qquad f_x = -3, \qquad f_y = -2.$$

The limits defining D are,

$$0 \le x \le 2, \qquad 0 \le y \le -\frac{3}{2}x + 3.$$

The surface are is then,

$$S = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{14}\, dy\, dx = \sqrt{14} \int_0^2 \left(-\frac{3}{2}x + 3 \right) dx =$$

$$= \sqrt{14} \left[-\frac{3x^2}{4} + 3x \right]_0^2 = 3\sqrt{14}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 Determine the surface area of the part of $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.

Solution $z = f(x, y) = xy$

Here are the partial derivatives $f_x = y, \;\; f_y = x$

The integral for the surface area is, $S = \iint_D \sqrt{x^2 + y^2 + 1}\,dA$

Given that D is a disk it makes sense to do this integral in polar coordinates

$$D = \{(r, \theta): \;\; 0 \le \theta \le 2\pi, \;\; 0 \le r \le 1\}.$$

$$S = \iint_D \sqrt{x^2 + y^2 + 1}\,dA = \int_0^{2\pi}\int_0^1 r\sqrt{r^2 + 1}\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}\left(\frac{2}{3}\right)\left(1 + r^2\right)^{\frac{3}{2}} \right]_0^1 d\theta =$$

$$= \int_0^{2\pi} \frac{1}{3}\left(2^{\frac{3}{2}} - 1\right) d\theta = \frac{2\pi}{3}\left(2^{\frac{3}{2}} - 1\right)$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Applications to physics:

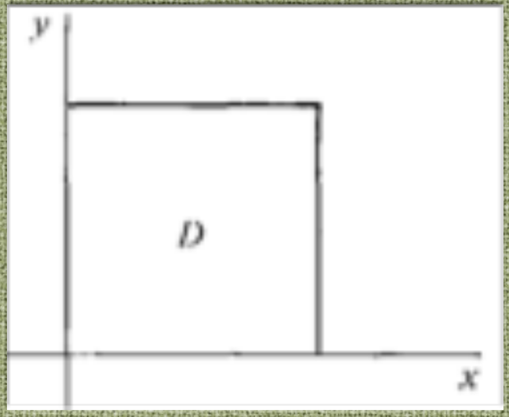
Mass, center of mass, and moment of inertia.

If a plane object fills a region D and has continuous density $\rho(x, y)$, its mass is $m = \iint_D \rho(x, y)\,dA$

Example 1 Find the mass of an object in the shape of a unit square whose density is the sum of the distance to the x - axis and twice the distance to the y - axis.

Solution

The region $D = \{(x, y): 0 \le x \le 1,\ \ 0 \le y \le 1\};$



The density $\rho(x, y) = y + 2x,$

$$m = \iint_D y + 2x\,dA = \int_0^1\int_0^1 y + 2x\,dy\,dx$$

$$\int_0^1 y + 2x\,dy = \left[\frac{y^2}{2} + 2xy\right]_0^1 = \frac{1}{2} + 2x$$

$$m = \int_0^1 \frac{1}{2} + 2x\,dx = \left[\frac{1}{2}x + x^2\right]_0^1 = \frac{3}{2}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

A plane object fills a region D and has continuous density $\rho(x,y)$ has ***moments*** about the x and the y axes given by $M_x = \iint_D y\rho(x,y)\,dA$

$$M_x = \iint_D y\rho(x,y)\,dA \qquad M_y = \iint_D x\rho(x,y)\,dA$$

M_x and M_y are sometimes called first moments to distinguish them from moments of inertia (which are called second moments).

The center of mass of the object is the point $(\bar{x},\bar{y})$ with coordinates

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} \qquad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

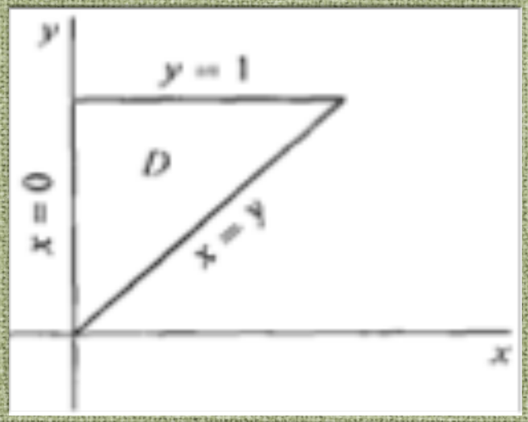
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 A triangle plate bounded by the lines $x = 0, x = y, \; y = 1$ has density $\rho(x, y) = x + y$. Find the moments and center of mass.

Solution

Sketch the region D

$D = \{0 \le x \le 1, \qquad 0 \le y \le 1.$



Setup and evaluate the interated integrals for the mass m and moments M_x and M_y.

$$m = \iint_D \rho(x, y)dA = \int_0^1 \int_x^1 x + ydydx$$

$$\int_x^1 x + ydy = [xy + \frac{y^2}{2}]_x^1 =$$

$$(x + \frac{1}{2}) - (x^2 + \frac{x^2}{2}) = x + \frac{1}{2} - \frac{3x^2}{2}$$

$$m = \int_0^1 x + \frac{1}{2} - \frac{3x^2}{2} dx = [\frac{x^2}{2} + \frac{1}{2}x - \frac{x^3}{2}]_0^1 = \frac{1}{2}$$

$$M_x = \iint_D y\rho(x, y)dA = \int_0^1 \int_x^1 y(x + y)dydx$$

$$\int_x^1 xy + y^2 dy = [x\frac{y^2}{2} + \frac{y^3}{3}]_x^1 =$$

$$(\frac{x}{2} + \frac{1}{3}) - (\frac{x^3}{2} + \frac{x^3}{3}) =$$

$$\frac{x}{2} + \frac{1}{3} - \frac{5x^3}{6}$$

$$M_x = \int_0^1 (\frac{x}{2} + \frac{1}{3} - \frac{5x^3}{6})dx = [\frac{x}{2} + \frac{1}{3} - \frac{5x^3}{6}]_0^1 = \frac{9}{24}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_x^1 x^2 + xy\,dy\,dx$$

$$\int_x^1 x^2 + xy\,dy = [x^2y + x\frac{y^2}{2}]_x^1 =$$

$$(x^2 + \frac{x}{2}) - (x^3 + \frac{x^3}{2}) =$$

$$x^2 + \frac{x}{2} - \frac{3x^3}{2}$$

$$M_y = \int_0^1 x^2 + \frac{x}{2} - \frac{3x^3}{2}\,dx = [\frac{x^3}{3} + \frac{x^2}{4} - \frac{3x^4}{8}]_0^1 = \frac{5}{24}$$

$$M_x = \frac{9}{24} \qquad M_y = \frac{5}{24}$$

$$\overline{x} = \frac{M_y}{m} = \frac{\frac{5}{24}}{\frac{1}{2}} = \frac{5}{12} \qquad \overline{y} = \frac{M_x}{m} = \frac{\frac{9}{24}}{\frac{1}{2}} = \frac{3}{4}$$

Thus the center of mass of the given object is the point $(\overline{x}, \overline{y}) = \left(\frac{5}{12}, \frac{3}{4}\right)$.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

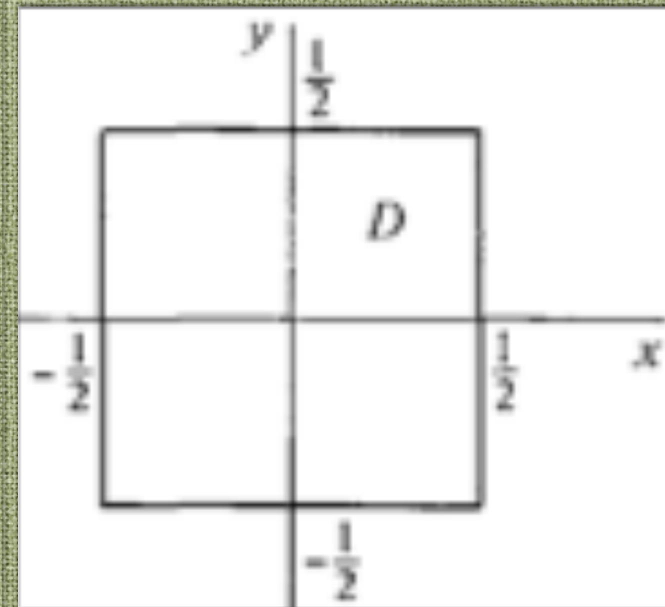
Given a plane object fills a region D and has continuous density $\rho(x,y)$ the ***moments of inertia*** about the origin is

$$I=\iint_D \rho(x,y)(x^2+y^2)\,dA$$

Example 4 Find the moment of inertia about the origin of an object with constant density $\rho=1$ which covers the square $-\frac{1}{2}\le x\le\frac{1}{2}, -\frac{1}{2}\le y\le\frac{1}{2}$.

Solution

$$I=\iint_D (x^2+y^2)\,dA=\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}(x^2+y^2)\,dy\,dx$$



$$\int_{-\frac{1}{2}}^{\frac{1}{2}}(x^2+y^2)\,dy=[x^2y+\frac{y^3}{3}]_{-\frac{1}{2}}^{\frac{1}{2}}=$$

$$(\frac{1}{2}x^2+\frac{(\frac{1}{2})^3}{3})-(-\frac{1}{2}x^2+\frac{(-\frac{1}{2})^3}{3})=x^2+\frac{1}{12}$$

$$I=\int_{-\frac{1}{2}}^{\frac{1}{2}}x^2+\frac{1}{12}\,dx=\frac{1}{6}$$

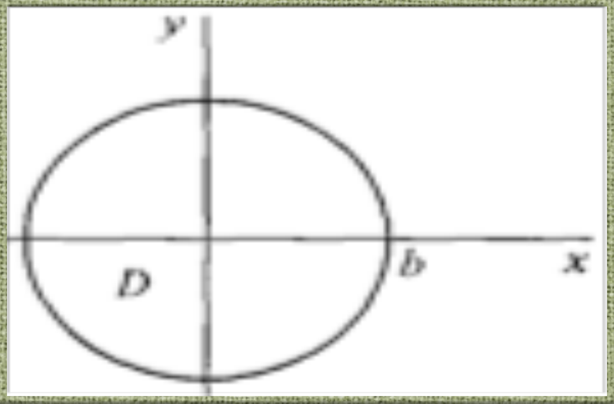
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 5 Find the moment of inertia about the origin of a circle centered at the origin with radius b and with constant density ρ.

Solution

Draw the region D



$$D=\{(r,\theta):\ 0\le\theta\le 2\pi,\ 0\le r\le b\}.$$

$x^2+y^2=r^2$, so

$$I=\iint_D \rho(x^2+y^2)\,dA=\int_0^{2\pi}\int_0^b \rho r^2 r\,dr\,d\theta$$

$$=\rho\int_0^{2\pi}\int_0^b r^3\,dr\,d\theta=\rho\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^b d\theta=\rho\int_0^{2\pi}\frac{b^4}{4}\,d\theta$$

$$=\frac{\rho b^4\pi}{2}$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Triple integrals

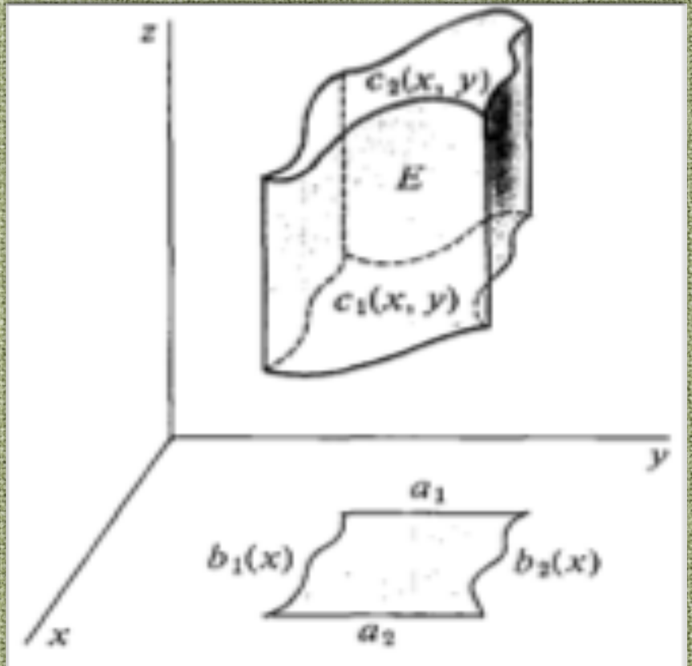
A closed region in space, or solid region, is a set E of points given by inequalities

$$a_1 \leq x \leq a_2, \qquad b_1(x) \leq y \leq b_2(x), \qquad c_1(x,y) \leq z \leq c_2(x,y),$$

where the functions $b_1(x), b_2(x)$ and $c_1(x,y), c_2(x,y)$ are continuous.

The *boundary* of E is the part of E on the following surfaces:

The planes $x = a_1$, $x = a_2$.

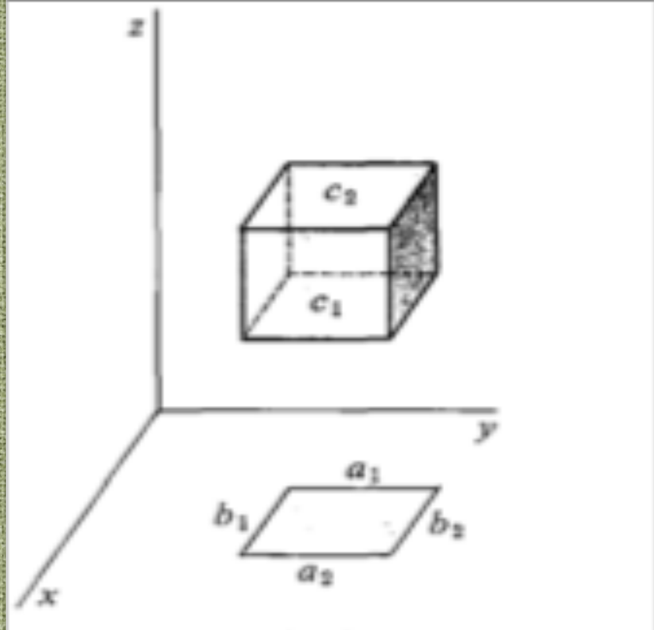


Region in space

The cylinders $y = b_1(x)$, $y = b_2(x)$.
The surfaces $z = c_1(x,y)$, $z = c_2(x,y)$.

The simplest type of closed region is a rectangular solid, or rectangular box,

$$a_1 \leq x \leq a_2, \qquad b_1 \leq y \leq b_2, \qquad c_1 \leq z \leq c_2.$$



Rectangular box

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Whenever we refer to a function $f(x,y,z)$ and a solid region E, we assume that $f(x,y,z)$ is continuous on some open region containing E. The triple integral of $f(x,y,z)$ over E is denoted by

$$\iiint_E f(x,y,z)\,dV$$

Iterated Integral Theorem

If E is the region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x,y) \le z \le c_2(x,y),$$

then

$$\iiint_E f(x,y,z)\,dV = \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)}\int_{c_1(x,y)}^{c_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If the region E is a rectangular box,

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2,$$

there are six different iterated integrals over E, corresponding to six different orders integration. Here they are

(1) $\displaystyle\int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dy\,dx$

(2) $\displaystyle\int_{a_1}^{a_2}\int_{c_1}^{c_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dz dx$

(3) $\displaystyle\int_{b_1}^{b_2}\int_{a_1}^{a_2}\int_{c_1}^{c_2} f(x,y,z)\,dz\,dx\,dy$

(4) $\displaystyle\int_{b_1}^{b_2}\int_{c_1}^{c_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dz dy$

(5) $\displaystyle\int_{c_1}^{c_2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y,z)\,dy\,dx\,dz$

(6) $\displaystyle\int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy dz$

The Iterated Integral Theore shows that each of these six iterated integrals is

equal to the triple integral $\displaystyle\iiint_E f(x,y,z)\,dV$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 1 Evaluate $\iiint_E xy^2z^3\,dV$ where E is the rectangular box

$$0 \le x \le 2, \qquad 0 \le y \le 1, \qquad 0 \le z \le 4.$$

First Solution $\iiint_E xy^2z^3\,dV = \int_0^2 \int_0^1 \int_0^4 xy^2z^3 dz\ dy\,dx$

The inside integral is

$$\int_0^4 xy^2z^3\,dz = \left[\frac{xy^2z^4}{4}\right]_0^4 = 64xy^2.$$

The second integral is

$$\int_0^1 64xy^2\,dy = \left[\frac{64xy^3}{3}\right]_0^1 = \frac{64x}{3}.$$

The final answer is

$$\int_0^2 \frac{64x}{3}\,dx = \left[\frac{64x^2}{6}\right]_0^2 = \frac{256}{6} = \frac{128}{3}\ .$$

Second Solution $\iiint_E xy^2z^3\,dV = \int_0^4 \int_0^2 \int_0^1 xy^2z^3 dy\ dx\,dz$

The inside integral is $\int_0^1 xy^2z^3\,dy = \left[\frac{xy^3z^3}{3}\right]_0^1 = \frac{xz^3}{3}.$

The second integral is $\int_0^2 \frac{xz^3}{3}\,dx = \left[\frac{x^2z^3}{6}\right]_0^2 = \frac{4z^3}{6} = \frac{2z^3}{3}.$

The final answer is

$$\int_0^4 \frac{2z^3}{3}\,dz = \left[\frac{2z^4}{12}\right]_0^4 = \left[\frac{z^4}{6}\right]_0^4 = \frac{256}{6} = \frac{128}{3}\ .$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 Evaluate $\iiint_E y+z\,dV$ where E is the region

$$0 \le x \le \frac{\pi}{2}, \qquad 0 \le y \le \sin x, \qquad 0 \le z \le y\cos x.$$

Solution $\displaystyle\iiint_E y+z\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\sin x}\int_0^{y\cos x} y+z\,dz\ dy\,dx$

We first evaluate the inside integral

$$\int_0^{y\cos x} y+z\,dz = \left[yz+\frac{z^2}{2}\right]_0^{y\cos x} = y^2\cos x + \frac{1}{2}y^2\cos^2 x.$$

Now we evaluate the second integral

$$\int_0^{\sin x} y^2\cos x + \frac{1}{2}y^2\cos^2 x\,dy = \left[\frac{y^3}{3}\cos x + \frac{1}{6}y^3\cos^2 x\right]_0^{\sin x} = \frac{\sin^3 x}{3}\cos x + \frac{\sin^3 x}{6}\cos^2 x.$$

Finally we evaluate the outside integral

$$\iiint_E y+z\,dV = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{3}\cos x + \frac{\sin^3 x}{6}\cos^2 x\,dx = \int_0^{\frac{\pi}{2}}\left[\frac{\sin^2 x}{3}\cos x + \frac{\sin^2 x}{6}\cos^2 x\right]\sin x\,dx$$

$$= -\int_0^{\frac{\pi}{2}}\left[\frac{(1-\cos^2 x)}{3}\cos x + \frac{(1-\cos^2 x)}{6}\cos^2 x\right](-\sin x\,dx) = -\int_1^0\left[\frac{(1-u^2)}{3}u + \frac{(1-u^2)}{6}u^2\right]du =$$

$$= \int_0^1\left[\frac{1}{3}(u-u^3)+\frac{1}{6}(u^2-u^4)\right]du = \left[\frac{1}{3}\left(\frac{u^2}{2}-\frac{u^4}{4}\right)+\frac{1}{6}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)\right]_0^1 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{6}\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{19}{180}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

The volume V of a region E in space is equal to the triple integral of the constant 1 over E as

$$V = \iiint_E dV\,.$$

The ***mass*** of an object filling a solid region E with continuous density $\rho(x,y,z)$ is

$$m = \iiint_E \rho(x,y,z)\,dV.$$

If an object in space fills a region E and continuous density $\rho(x,y,z)$, its moments about the cordinate planes are

$$M_{xy} = \iiint_E z\rho(x,y,z)\,dV,$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV,$$

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV.$$

The ***center of mass*** of the object is the point $(\bar{x},\bar{y},\bar{z})$, where m is mass and

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

If an object in space fills a region E and continuous density $\rho(x,y,z)$, its ***moments of inertia*** about the cordinate axes are

$$I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV,$$

$$I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV,$$

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV.$$

Example 4 An object has constant density and the shape of a tetrahedron with vertices at the four points

$$(0,0,0), \qquad (1,0,0), \qquad (0,1,0), \qquad (0,0,1).$$

Find the center of mass.

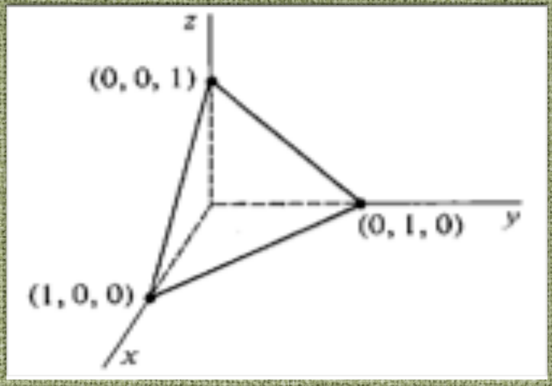
Remark When the density is constant, the center of mass is called ***centroid***.

Solution The region E is the solid bounded by the coordinate planes and the plane $x+y+z=1$ which passes through $(1,0,0), (0,1,0), (0,0,1)$.
Solving for z, the plane is

$$z = 1-x-y.$$

This plane meets the plane $z=0$ at the line $1-x-y=0$, or $y=1-x$.
Therefore the Therefore E is the region

$$0 \le x \le 1, \qquad 0 \le y \le 1-x, \qquad 0 \le z \le 1-x-y.$$



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Let the density be $\rho = 1$,

$$m = \iiint_E dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 \int_0^{1-x} 1 - x - y\ \ dy\, dx$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx = \frac{1}{2}\int_0^1 (1-x)^2\ dx$$

$$= \frac{1}{2}\left[\frac{-(1-x)^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3}\right] = \frac{1}{6}.$$

$$M_{yz} = \iiint_E x dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz\ dy\ dx = \int_0^1 \int_0^{1-x} x - x^2 - xy\ \ dy\ dx$$

$$\frac{1}{2}\int_0^1 x(1-x)^2\ dx = \frac{1}{24}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{\frac{1}{24}}{\frac{1}{6}} = \frac{1}{4}\ .$$

Similarly, we obtain $\bar{y} = \frac{1}{4}, \bar{z} = \frac{1}{4}$. Then, the centroid is $(\bar{x}, \bar{y},\ \bar{z}) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

Prof. Messaoud Boumkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 5 Find the moments of inertia about the three axes of an object with constant density 1 filling the cube

$$0 \leq x \leq a, \qquad 0 \leq y \leq a, \qquad 0 \leq z \leq a.$$

Solution

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV$$

$$= \iiint_E (y^2 + z^2)\, dV$$

$$= \int_0^a \int_0^a \int_0^a (y^2 + z^2) dz\, dy\, dx$$

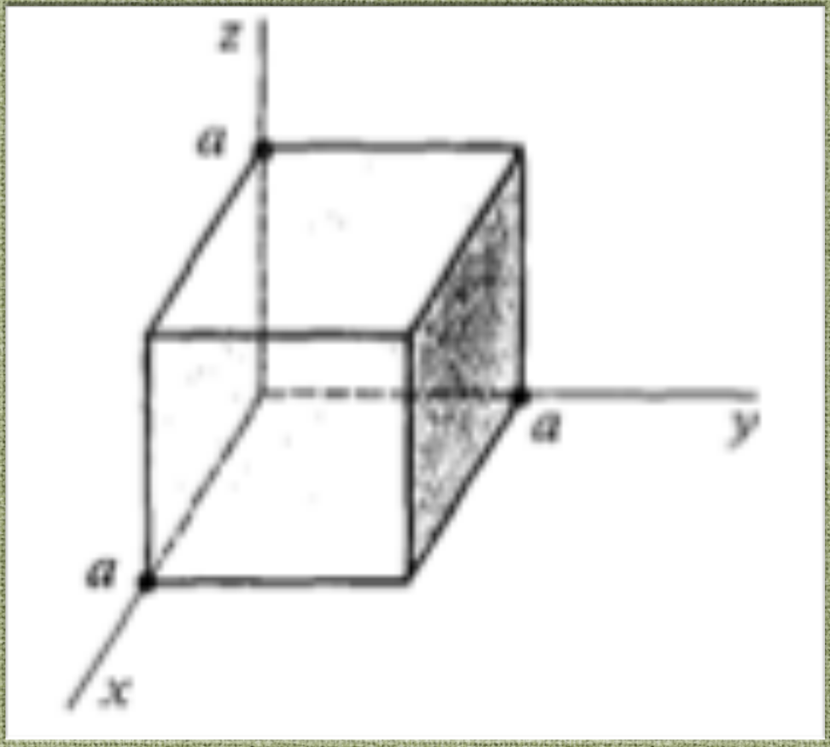
$$= \int_0^a \int_0^a \left[zy^2 + \frac{z^3}{3} \right]_0^a dy\, dx$$

$$= \int_0^a \int_0^a \left(ay^2 + \frac{a^3}{3} \right) dy\, dx$$

$$= \int_0^a \left[\frac{ay^3}{3} + \frac{a^3}{3} y \right]_0^a dx = \int_0^a \frac{2a^4}{3} dx = \frac{2a^5}{3}.$$

Similarly, we obtain

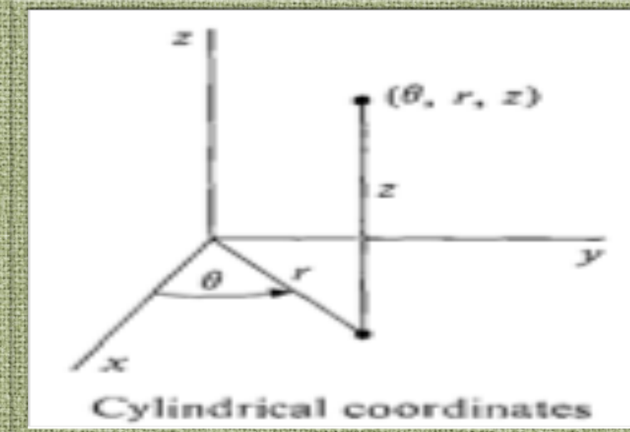
$$I_y = \frac{2a^5}{3}, \qquad I_z = \frac{2a^5}{3}.$$



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Cylindrical and spherical coordinates



Cylindrical coordinates

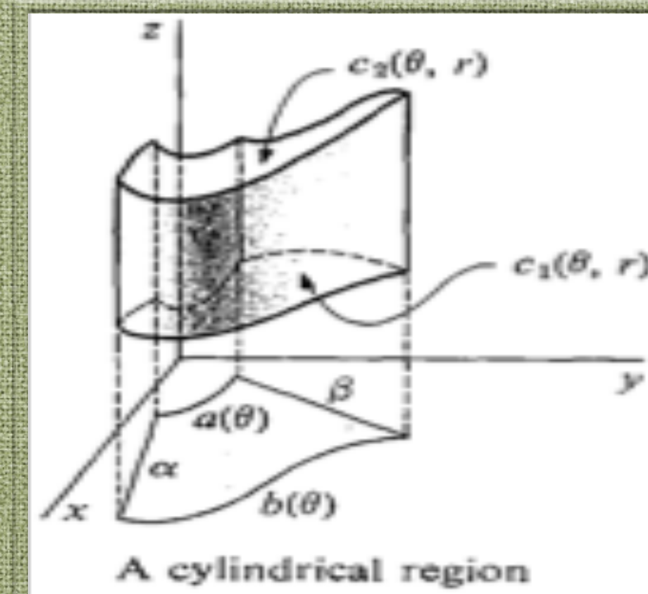
In evaluating triple integrals it is sometimes easier to use cylindrical or spherical coordinates instead of rectngular coordinates.

A point (x, y, z) has cylindrical coordinates (θ, r, z) if

$$x = r\cos\theta, y = r\sin\theta, z = z,$$

where

$$0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le r.$$



A cylindrical region

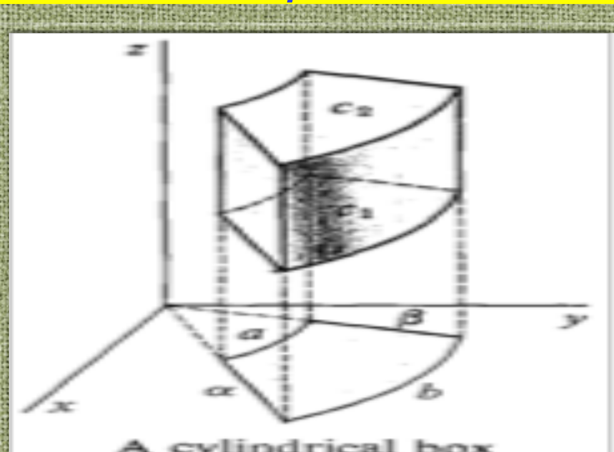
A ***cylindrical region*** is a region E in (x, y, z) space given by cylindrical coordinate inequalities,

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r),$$

$$0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le r.$$

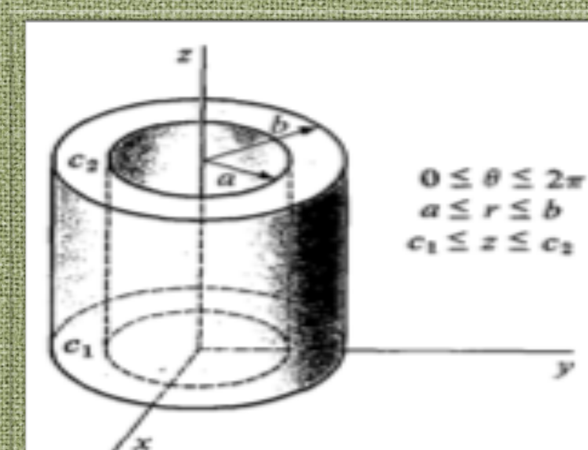
The simplest kind of cylindrical region is the *cylindrical box*

$$\alpha \le \theta \le \beta, \qquad a \le r \le b, \qquad c_1 \le z \le c_2,$$



A cylindrical box

If $0 \le \theta \le 2\pi$, the *cylindrical box* has the form



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Let E be the cylindrical region

$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r).$

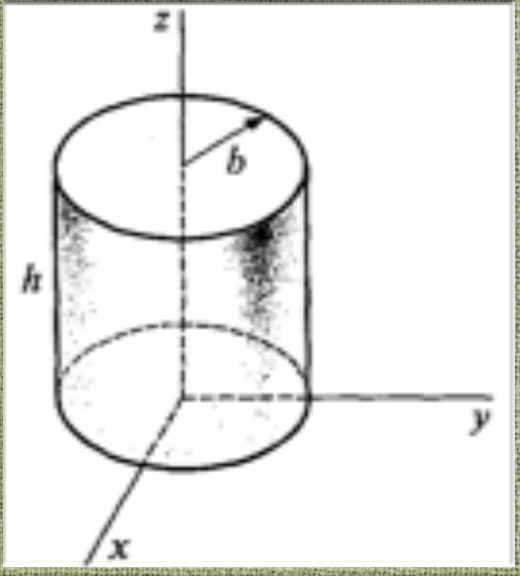
The triple integral of $f(x, y, z)$ over E is

$$\iiint_E f(x,y,z)\, dV = \int_{\alpha}^{\beta} \int_{a(\theta)}^{b(\theta)} \int_{c_1(\theta,r)}^{c_2(\theta,r)} f(r\cos\theta, r\sin\theta, z) r dz\; dr\; d\theta.$$

Example 1 Find the moment of inertia of a cylinder of height h, base a circle of radius b, and constant density 1, about its axe.

Solution Draw the region and write the given cylinder in cylindrical coordinates as

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad 0 \le z \le h.$$



$$I_z = \iiint_E x^2 + y^2\, dV = \int_0^{2\pi} \int_0^b \int_0^h r^2 r dz\; dr\; d\theta$$

$$= \int_0^{2\pi} \int_0^b \int_0^h r^3 dz\; dr\; d\theta = \int_0^{2\pi} \int_0^b r^3 h\; dr\; d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} h\right]_0^b d\theta = \int_0^{2\pi} \frac{hb^4}{4}\, d\theta = \frac{2\pi hb^4}{4} = \frac{\pi hb^4}{2}.$$

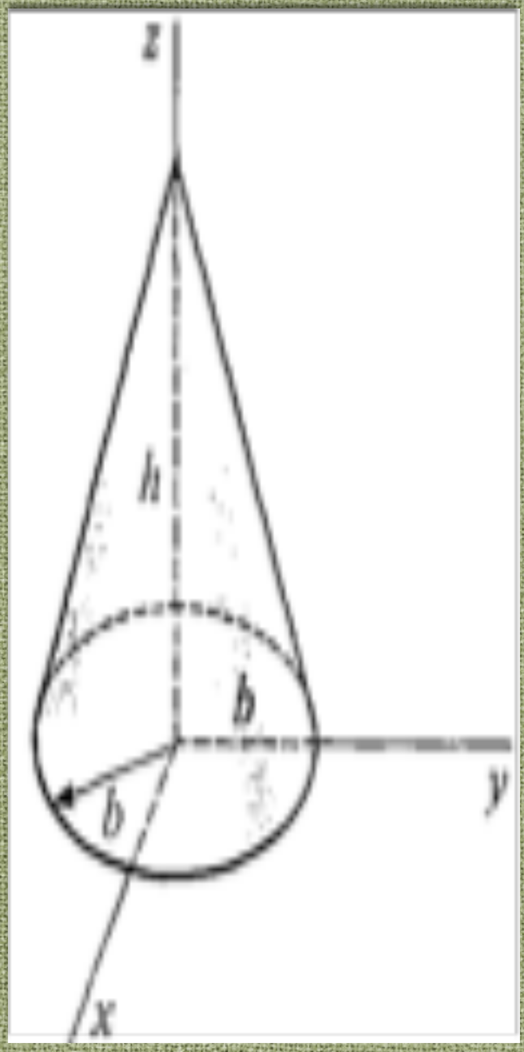
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 2 Find the centroid of the positive part of the cone $z = h - \frac{h}{b}\sqrt{(x^2 + y^2)}$.

Solution Draw the region and write the given solid in cylindrical coordinates as

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad 0 \le z \le h - \frac{hr}{b}.$$



Let the density be 1.

$$m = \iiint_E dV = \int_0^{2\pi}\int_0^{b}\int_0^{h-\frac{hr}{b}} rdz\ dr\,d\theta = \int_0^{2\pi}\int_0^{b}\left(h - \frac{hr}{b}\right) r\ dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{b} hr - \frac{hr^2}{b}\ dr\,d\theta = \int_0^{2\pi}\left[\frac{hr^2}{2} - \frac{hr^3}{3b}\right]_0^b d\theta$$

$$= \int_0^{2\pi} \frac{hb^2}{2} - \frac{hb^3}{3b}\ d\theta = \int_0^{2\pi} \frac{hb^2}{2} - \frac{hb^2}{3}\ d\theta = \int_0^{2\pi} \frac{hb^2}{6}\ d\theta = \frac{2\pi hb^2}{6}$$

$$= \frac{\pi hb^2}{3}.$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

$$M_{xy} = \iiint_E z dV = \int_0^{2\pi}\int_0^b\int_0^{h-\frac{hr}{b}} zrdz\ dr\ d\theta = \int_0^{2\pi}\int_0^b \frac{1}{2}\left(h - \frac{hr}{b}\right)^2 r\ dr\ d\theta$$

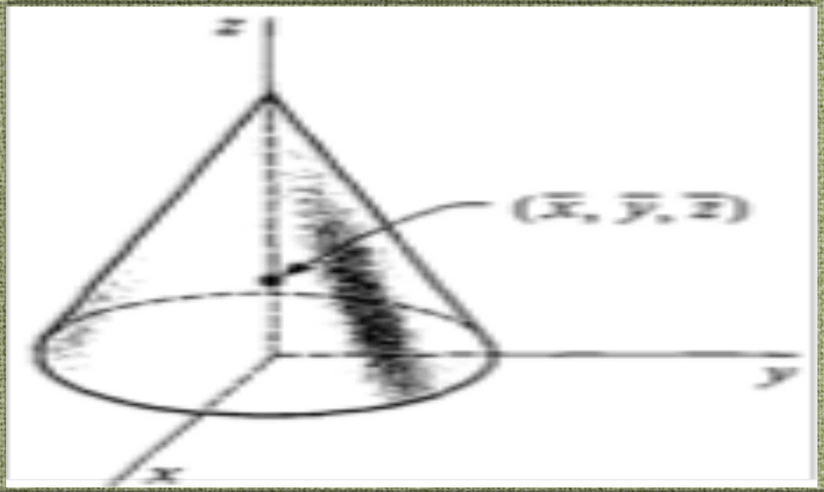
$$= \frac{h^2}{2}\int_0^{2\pi}\int_0^b r - \frac{2r^2}{b} + \frac{r^3}{b^2} dr\ d\theta = \frac{h^2}{2}\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{2r^3}{3b} + \frac{r^4}{4b^2}\right]_0^b d\theta$$

$$= \frac{h^2}{2}\int_0^{2\pi} \frac{b^2}{2} - \frac{2b^2}{3} + \frac{b^2}{4}\ d\theta = \frac{h^2}{2}\int_0^{2\pi}\frac{b^2}{12}\ d\theta = \frac{h^2}{2}\frac{2\pi b^2}{12} = \frac{\pi h^2 b^2}{12}.$$

Since the cone is symetric about the $z-$axis, so $\bar{x} = 0$ *and* $\bar{y} = 0$.

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\frac{\pi h^2 b^2}{12}}{\frac{\pi h b^2}{3}} = \frac{h}{4}.$$

Then the centroid $(\bar{x}, \bar{y}, \bar{z}) = \left(0,0,\frac{h}{4}\right)$.



Spherical Coordinates

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

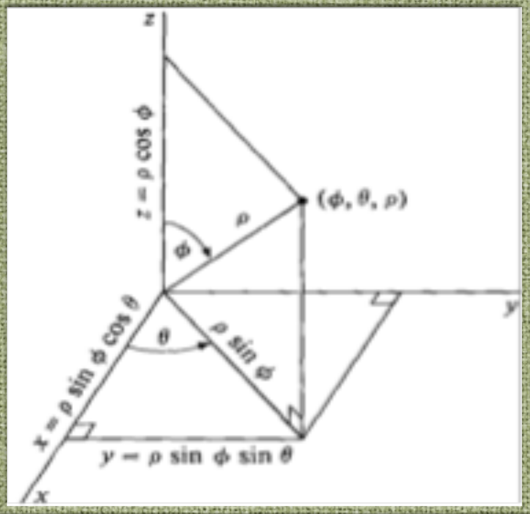
To express a point $P(x, y, z)$ in *spherical coordinates* we let ρ be the distance from the origin to P, let θ be the same angle as in *cylindrical coordinates*, and let ϕ be the angle between the positive $z-$axis and the line OP. Note that ϕ can always be chosen between 0 and π.

A point (x, y, z) has *spherical coordinates* (ϕ, θ, ρ) if

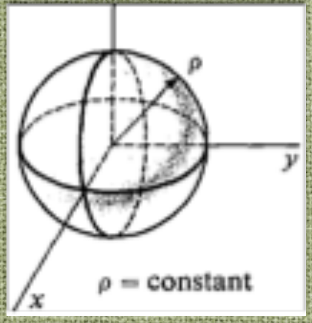
$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi,$$

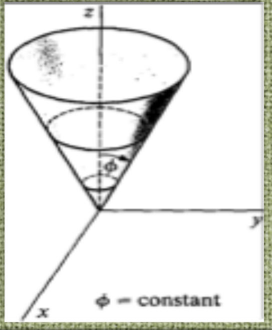
where

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi, \quad \text{and} \quad 0 \leq \rho.$$

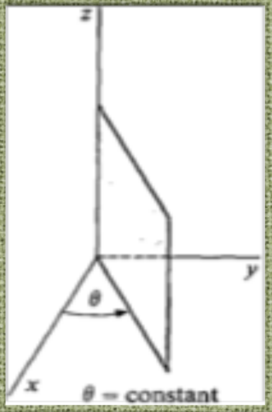


The graph of the equation $\rho = \text{constant}$ is a sphere with center at the origin.





The graph of the equation $\phi = \text{constant}$ is a vertical cone with vertex at the origin.



The graph of the equation $\theta = \text{constant}$ is a half $-$ plane through the $z-$axis.

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

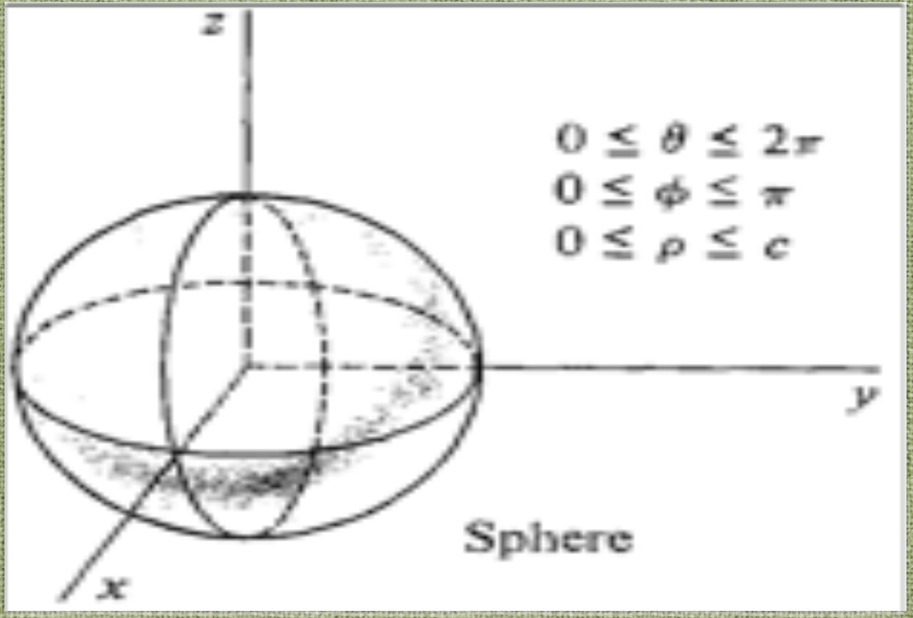
A ***spherical region*** is a region E in (x, y, z) space given by spherical coordinate inequalities,

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1(\theta) \le \phi \le \beta_2(\theta), \qquad c_1(\theta, \phi) \le \rho \le c_2(\theta, \phi),$$

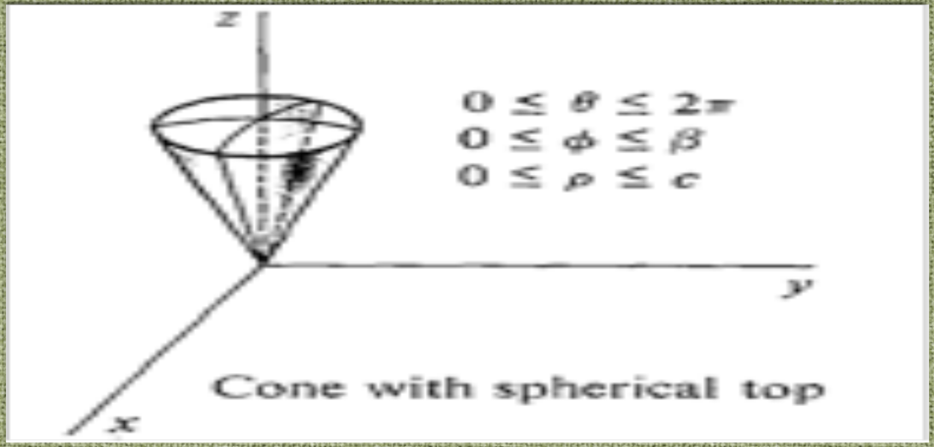
$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi, \text{ and } \quad 0 \le \rho,$$

where all the functions are continuous.

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le c.$$



$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \beta, \qquad 0 \le \rho \le c.$$



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Let E be a spherical region

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1(\theta) \le \phi \le \beta_2(\theta), \qquad c_1(\theta,\phi) \le \rho \le c_2(\theta,\phi).$$

The triple integral of $f(x,y,z)$ over E is

$$\iiint_E f(x,y,z)\, dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\theta,\phi)}^{c_2(\theta,\phi)} f(x,y,z)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= \int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\theta,\phi)}^{c_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

The triple integral for volume $V = \iiint_E dV$,

gives us iterated integral formulas for volume in rectangular, cylindrical, and spherical coordinates

Rectangular $\quad V = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} dz \, dy \, dx$

Cylindrical $\quad V = \int_{\alpha}^{\beta} \int_{a(\theta)}^{b(\theta)} \int_{c_1(\theta,r)}^{c_2(\theta,r)} r dz \, dr \, d\theta$

Spherical $\quad V = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\theta,\phi)}^{c_2(\theta,\phi)} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

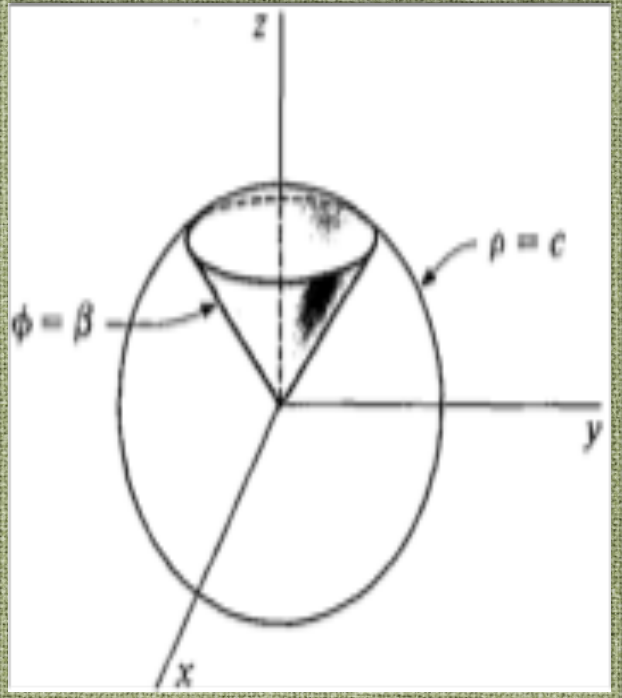
Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 3 Find the volume of the region above the cone $\phi = \beta$ and inside the sphere $\rho = c$.

Solution The region is given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \beta, \qquad 0 \le \rho \le c.$$



Then

$$V = \int_0^{2\pi} \int_0^{\beta} \int_0^{c} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\beta} \frac{c^3}{3} \sin\phi \, d\phi \, d\theta$$
$$= \int_0^{2\pi} \frac{c^3}{3} [-\cos\phi]_0^{\beta} \, d\theta$$
$$= \int_0^{2\pi} \frac{c^3}{3} (1 - \cos\beta) \, d\theta = \frac{2\pi c^3}{3} (1 - \cos\beta).$$

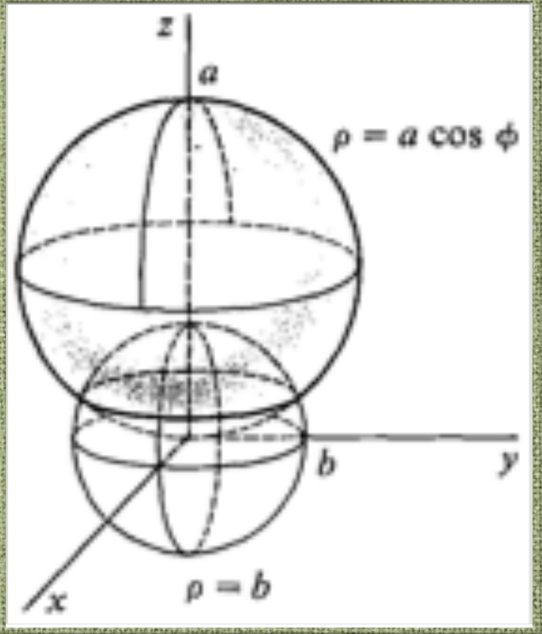
Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 4 A sphere of diameter a passes through the center of a sphere of radius b, and $a > b$. Find the volume of the region inside the sphere of diameter a and outside the sphere of radius b.

Solution The region is given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \cos^{-1}\left(\frac{b}{a}\right), \qquad b \le \rho \le a\cos\phi .$$



The two spheres $\rho = a\cos\phi$ and $\rho = b$. They intersect at $\phi = \cos^{-1}\left(\frac{b}{a}\right)$.

$$V = \int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{b}{a}\right)}\int_0^{a\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{b}{a}\right)} \left[\frac{\rho^3}{3}\sin\phi\right]_b^{a\cos\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{b}{a}\right)} \left(\frac{a^3\cos^3\phi}{3} - \frac{b^3}{3}\right)\sin\phi \, d\phi \, d\theta.$$

Put $u = \cos\phi$, $\quad du = -\sin\phi$. Then

$$V = -\frac{1}{3}\int_0^{2\pi}\int_1^{\frac{b}{a}} (a^3u^3 - b^3)\, du \, d\theta = -\frac{1}{3}\int_0^{2\pi}\left[\frac{a^3u^4}{4} - ub^3\right]_1^{\frac{b}{a}} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left(\frac{a^3\left(\frac{b}{a}\right)^4}{4} - \frac{b}{a}b^3\right) - \left(\frac{a^3}{4} - b^3\right) d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\frac{3b^4}{4a} - b^3 + \frac{a^3}{4}\, d\theta = \frac{\pi}{6}\left(\frac{3b^4}{a} - 4b^3 + a^3\right).$$

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Example 5 Find the mass of a sphere of radius c whose density is equal to the distance from the surface.

Solution The region is given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le c.$$

The density at (θ, ϕ, ρ) is density $= c - \rho$.

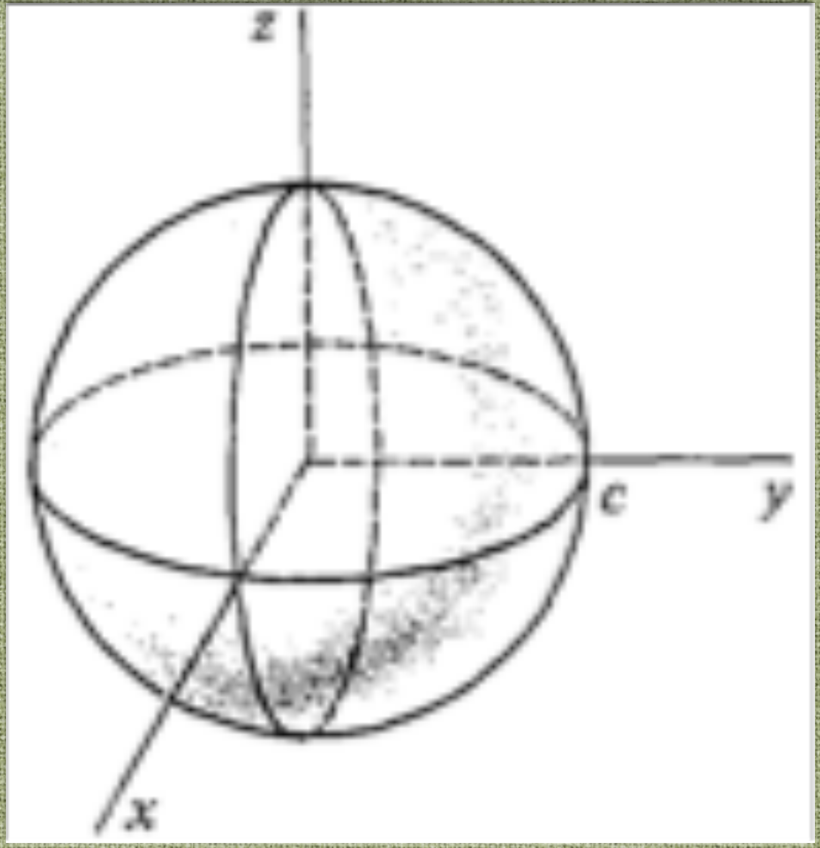
The mass is

$$m = \int_0^{2\pi} \int_0^{\pi} \int_0^{c} (c-\rho)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{c} c\rho^2 - \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{c\rho^3}{3} - \frac{\rho^4}{4}\right]_0^c \sin\phi \, d\phi \, d\theta = \frac{c^4}{12} \int_0^{2\pi} [-\cos\phi]_0^{\pi} \, d\theta$$

$$= \frac{c^4}{12} \int_0^{2\pi} (-\cos\pi + \cos 0) \, d\theta = \frac{4\pi c^4}{12} = \frac{\pi c^4}{3}.$$



Prof. Messaoud Bounkhel
Department of Mathematics
King Saud University

Problems on Chapter 2:

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

In Problems 1-6, evaluate the integral using cylindrical coordinates.

1. $V = \iiint_E z\sqrt{x^2+y^2}\,dV$, E is the cylinder $x^2+y^2 \le 1,\ \ 0 \le z \le 2$.
2. $V = \iiint_E x^2 + z dV$, E is the cylinder $x^2+y^2 \le 9, 0 \le z \le 6$.
3. $V = \iiint_E x^2 + y^2 dV$, E is the cone $x^2+y^2 \le 1,\ \ 0 \le z \le 1-\sqrt{x^2+y^2}$.
4. $V = \iiint_E 4+\sqrt{z} dV$, E is the cone $x^2+y^2 \le 1, \sqrt{x^2+y^2} \le z \le 1$.
5. $V = \iiint_E (x+y)z dV$, E is the region $0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}, 0 \le z \le x^2+y^2$.
6. $V = \iiint_E \frac{z}{\sqrt{x^2+y^2}} dV$, E is the region $0 \le x^2+y^2 \le 4, 0 \le z \le |x|$.
7. Find the mass of an object in the shape of a cylinder of radius b and height h whose density is equal to the distance from the axis.
8. Find the mass of an object in the shape of a cylinder of radius b and height h whose density is equal to the distance from the base.
9. Find the mass of an object in the shape of a cone of radius b and height h whose density is equal to the square of the distance from the axis.
10. Find the mass of an object in the shape of a cone of radius b and height h whose density is equal to the sum of the distance from the base and of the distance from the axis.
11. Find the centroid of an object filling the region above the paraboloid $z = x^2+y^2$ and below the plane $z = 1$.

1. Find the centroid of an object filling the region $x^2+y^2 \le b, 0 \le z \le \sqrt{x^2+y^2}$.
2. Find the moment of inertia about the x -axis of an object of constant density k in the cylinder $0 \le r \le b,\ -c \le z \le c$.
1. Find the moment of inertia about the z-axis of an object of constant density k in the cylindrical shell $a \le r \le b,\ -c \le z \le c$.

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University

15. Find the moment of inertia of an object of constant density k in a cone of radius b and height h about its axis.

In Problems 16-24, evaluate the integral using spherical coordinates.

16. $V = \iiint_E x^2 + y^2 + z^2 dV$, E is the sphere $x^2 + y^2 + z^2 \leq b^2$.
17. $V = \iiint_E \sqrt{x^2 + y^2 + z^2}\, dV$, E is the sphere $x^2 + y^2 + z^2 \leq b^2$.
18. $V = \iiint_E x^2 dV$, E is the sphere $x^2 + y^2 + z^2 \leq 1$.
19. $V = \iiint_E z^2 dV$, E is the sphere $x^2 + y^2 + z^2 \leq 1$.
20. $V = \iiint_E z dV$, E is the sphere $\rho \leq 2b \cos\phi$.
21. $V = \iiint_E (x^2 + y^2 + z^2)^{\frac{3}{2}} dV$, E is the intersection of the spheres $\rho \leq 2b \cos\phi, \rho \leq b$.
22. $V = \iiint_E z(x^2 + y^2 + z^2)^{\frac{1}{2}} dV$, E is the region above the cone $\phi = \alpha$ and inside the sphere $\rho = b$.
23. $V = \iiint_E \frac{1}{x^2+y^2+z^2} dV$, E is the spherical shell $a \leq \rho \leq b$.
24. Find the volume of the spherical shell $a \leq \rho \leq b$.
25. Find the volume of the spherical box $\alpha_1 \leq \theta \leq \alpha_2, \beta_1 \leq \phi \leq \beta_2, c_1 \leq \rho \leq c_2$.
26. Find the volume of the region above the cone $\phi = \beta$ and inside the sphere $\rho = b \cos\phi$.
27. Find the volume of the spherical region $0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi,\ 0 \leq \rho \leq \sin\phi$.
28. Find the mass of an object in the shape of a sphere of radius c whose density is equal to the distance from the center.

16. Find the mass of a spherical shell $a \leq \rho \leq b$ whose density is equal to the reciprocal of the distance from the center.

16. Find the moment of inertia of a spherical object of radius b and constant density k about a diameter of the sphere.
17. Find the moment of inertia of a spherical shell $a \leq \rho \leq b$ of constant density k about any diameter.
18. Find the centroid of a hemisphere of radius b.

Prof. Messaoud Boukhel
Department of Mathematics
King Saud University