King Saud University
Department of Mathematics

244
First Midterm, November 2016
Solutions

NAME:

Group Number/Instructor name:

ID:

• Duration of the exam: 90 minutes

Question	Grade
I	5
II	3
III	5
IV	7
Total	

Question	1	2	3	4	5
Answer	a	b	b	a	c

I) Choose the correct answer (write it on the table above):

1) If $A^3 - 2B^{-1} = \begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, then the matrix A is

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$	(b) $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$	(c) $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$	(d) None

2) If $2A = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$ and $p(x) = x^2 + 1$, then $p(A)$ equals

(a) $\begin{bmatrix} 5 & 4 \\ 10 & 9 \end{bmatrix}$	(b) $\begin{bmatrix} 5 & 3 \\ 9 & 8 \end{bmatrix}$	(c) $\begin{bmatrix} -1 & 4 \\ 9 & 2 \end{bmatrix}$	(d) None

3) (1) If A, B and C are $n \times n$ matrices, then $AC - (C^T B)^T$ equals

(a) $(A - B)C^T$	(b) $(A - B^T)C$	(c) $(AC - BC)^T$	(d) None

4) For any $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, the system $\begin{cases} -x + 2y = b_1 \\ 2x + y = b_2 \end{cases}$ has

(a) unique solution	(b) infinitely many solution	(c) no solution	(d) None

5) If $A = \begin{bmatrix} k & 0 & 0 \\ -1 & k & 0 \\ 0 & 0 & k-5 \end{bmatrix}$, then A is invertible if and only if

(a) $k = 0$ or $k = 5$	(b) $k \neq 0$ or $k \neq 5$	(c) $k \neq 0$ and $k \neq 5$	(d) None

II) Determine whether the following is **True** or **False**.

(1) Every matrix has a unique row echelon form. **(F)**

(2) The product of two elementary matrices of the same size is an elementary matrix . **(F)**

(3) If A is $n \times n$ symmetric matrix, then $A + 4I_n$ is symmetric matrix. **(T)**

(4) If A is an upper triangular matrix, then the matrix $A - A^T$ is diagonal. **(F)**

(5) Any homogenous linear system is consistent. **(T)**

(6) If A and B are $n \times n$ matrices that commute, then $(A + B)^2 - 2AB = A^2 + B^2$. **(T)**

III) a) Solve the linear system of equations

$$\begin{cases} x + y + 2z = 0 \\ 2x + y + z = 1 \\ 3x + 2y + 5z = 2 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 5 & 2 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -3 & 1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \xrightarrow{2R_2} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_2+R_3}$$
$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow[-3R_3+R_2]{-2R_3+R_1} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow{-R_2+R_1}$$
$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$$x = \frac{3}{2},\ y = -\frac{5}{2}, \text{and } z = \frac{1}{2}.$$

b) Show that if A and B are invertible matrices with the same size, then AB is invertible are $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

Since A and B are invertible then
there exists A^{-1} *and* B^{-1} *s.t.* $AA^{-1} = A^{-1}A = I$ and $BB^{-1} = B^{-1}B = I$
We want to prove that AB is invertible and the inverse is $B^{-1}A^{-1}$ (i.e. $B^{-1}A^{-1}AB = ABB^{-1}A^{-1} = I$) ?!

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

similarly

$$ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

IV) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$

a) Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_2]{R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{-R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} & 1 \end{array}\right]$$

then $A^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix}$

b) If R is the reduced row echelon form of A, then find $(AR)^{-1}$.
Since A is invertible, then the reduced row echelon form of A is $R = I_3$

$$(AR)^{-1} = (AI_3)^{1} = A^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix}$$

c) Find A^{-T}.

$$A^{-T} = (A^{-1})^T = \begin{bmatrix} -1 & -1 & 2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix}$$

d) Without solving the system find the solution of $A^k\mathbf{x} = 0$ and justify your answer.
Since A is invertible, then A^k is invertible (multiple of invertible matrices is an invertible matrix). Using the equivalent statement, since A^k is an invertible matrix then the system $A^k\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$