King Saud University
Department of Mathematics

First Midterm Exam

2^{nd} **semester 1439**

Course Title: Math 316 (Mathematical Methods)

Date: Feb. 2019; (10-11:30) am

(. . . . . .) Name . . . . ID . . . .

Question	Grade
Q1	
Q2	
Q3	
Q4	
Total	

Question 1

Prove or disprove each of the following sentences:

(a) If $f(x) = \frac{1}{\sqrt{x}}$ and $\rho(x) = x$, then $f \notin \mathcal{L}^2_\rho(0,1)$.

$\|f\|^2_\rho = \int_0^1 |f|^2 x\ dx = \int_0^1 |\frac{1}{\sqrt{x}}|^2 x\ dx = x|_0^1 = 1 < \infty \implies f \in L^2_\rho(0,1)$

(b) If a set $\{x_1, x_2, ..., x_n\}$ is orthogonal in an inner product space X, then it is linearly independent.

To prove $\{x_1, x_2, ..., x_n\}$ is linearly independent we assume that $\sum_{i=1}^n a_i x_i = 0$, we need to prove that $a_i = 0 \ \ \forall i$

$$\langle \sum_{i=1}^n a_i x_i, x_j \rangle = \langle 0, x_j \rangle = 0$$

$$\sum_{i=1}^n a_i < x_i, x_j >= 0 \text{ but } < x_i, x_j >= \begin{cases} \|x_i\|^2 & i = j \\ 0 & otherwise \end{cases}$$

$$a_i \|x_i\|^2 = 0$$

$$a_i = 0 \quad \forall i$$

(c) In the space L^2, every convergent sequence is Cauchy sequence.

From the book

Question 2

Let $f_n(x) = x^n, \quad 0 \leq x \leq 1.$

(a) Calculate the pointwise limit f.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

(b) Is the convergence uniform? Justify your answer.

NO, let $0 < \epsilon < 1$

$$|f_n(x) - f(x)| = |x^n - 0| = x^n$$

so for any $N \in \mathbf{N}$

$$x^n < \epsilon \iff x < \sqrt[N]{\epsilon}$$

that is for all $x \in [\sqrt[N]{\epsilon}, 1)$

$$|x^N - 0| > \epsilon \implies f_n \overset{u}{\not\to} f$$

(c) Determine the convergence in $\mathcal{L}^2$.

Since $\mathcal{L}^2(0,1) = \mathcal{L}^2[0,1]$, then we could use the limit $f(x) = 0$

$$\|f_n(x) - f(x)\|^2 = \|x^n - 0\|^2 = \int_0^1 |x^n|^2 \, dx = \frac{x^{2n+1}}{2n+1}\Big|_0^1 = \frac{1}{2n+1}$$

but

$$\lim_{n\to\infty} \frac{1}{2n+1} = 0 \implies f_n(x) \xrightarrow{\mathcal{L}^2} 0$$

Question 3

Prove that the infinite sequence $\{1,\ \sin nx,\ \cos nx,\ n = 1, 2, 3, \dots\}$ is orthogonal in the real inner product space $\mathcal{L}^2(-\pi, \pi)$.

To prove that the set is orthogonal we have to prove that the inner product of each two members is zero.

$$\langle 1, \sin nx\rangle = \int_{-\pi}^{\pi} \sin nx\ dx = -\frac{1}{n}\cos nx|_{-\pi}^{\pi} = 0$$
$$\langle 1, \cos nx\rangle = \int_{-\pi}^{\pi} \cos nx\ dx = \frac{1}{n}\sin nx|_{-\pi}^{\pi} = 0$$

$$\begin{aligned}\langle \sin nx, \sin mx\rangle &= \int_{-\pi}^{\pi} \sin nx \sin mx\ dx \\ &= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(n-m)x - \cos(n+m)x]\ dx \\ &= \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} = 0\end{aligned}$$

$$\begin{aligned}\langle \cos nx, \cos mx\rangle &= \int_{-\pi}^{\pi} \cos nx \cos mx\ dx \\ &= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(n+m)x + \cos(n-m)x]\ dx \\ &= \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} + \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} = 0\end{aligned}$$

$$\begin{aligned}\langle \sin nx, \cos mx\rangle &= \int_{-\pi}^{\pi} \sin nx \cos mx\ dx \\ &= \int_{-\pi}^{\pi} \frac{1}{2}[\sin(n+m)x + \sin(n-m)x]\ dx \\ &= \frac{1}{2}\left[\frac{\cos(n-m)x}{n-m} + \frac{\cos(n+m)x}{n+m}\right]_{-\pi}^{\pi} = 0\end{aligned}$$

Question 4

I. If X is an inner product space, then

$$\|x+y\| \leq \|x\| + \|y\|$$

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, \overline{x+y}\rangle \\ &=< x,x > + < x,y > + < y,x > + < y,y > \\ &= \|x\|^2 + < x,y > + \overline{< x,y >} + \|y\|^2 \\ &= \|x\|^2 + 2Re < x,y > + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= [\|x\| + \|y\|]^2 \end{aligned}$$

II. Determine the real values of α for which x^α lies in $\mathcal{L}^2(0,1)$

$$\|x^\alpha\|^2 = \int_0^1 x^{2\alpha}\, dx = \frac{x^{2\alpha+1}}{2\alpha+1}|_0^1 = \frac{1}{2\alpha+1} < \infty$$

which implies that $2\alpha + 1 \neq 0 \implies \alpha \neq -\frac{1}{2}$