King Saud University
Department of Mathematics

Second Midterm Exam

2^{nd} **semester 1439**

Course Title: Math 316 (Mathematical Methods)

Date: Mar. 2019; (10-11:30) pm

(. . . . . .) Name . . . . ID . . . .

Question	Grade
Q1	
Q2	
Q3	
Q4	
Total	

Question 1

Prove each of the following sentences:

(i) If y_1 and y_2 are linearly independent solutions of $y'' + q(x)y' + r(x)y = 0$, then $W(y_1, y_2) \neq 0$.

Solution

Assume that y_1 and y_2 are linearly dependent $\implies$ $y_1 = cy_2$ where c is a constant
We find $W(y_1, y_2)$

$$\begin{aligned} W(y_1, y_2) &= \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \det \begin{vmatrix} cy_2 & y_2 \\ cy_2' & y_2' \end{vmatrix} \\ &= cy_2y_2' - cy_2y_2' \\ &= 0 \end{aligned}$$

If y_1 and y_2 are linearly dependent solutions of $y'' + q(x)y' + r(x)y = 0$, then $W(y_1, y_2) = 0$

which implies

If y_1 and y_2 are linearly independent solutions of $y'' + q(x)y' + r(x)y = 0$, then $W(y_1, y_2) \neq 0$

(ii) If

$$x = \sum_{n=1}^{\infty} a_n \sin nx$$

where $0 \leq x \leq \pi$, then $a_n = 2\frac{(-1)^{n+1}}{n}$.

Solution
Since that the set $\{\sin nx, n \in \mathbf{N}\}$ is an orthogonal set, then

$$\begin{aligned} a_n &= \frac{\langle x, \sin nx \rangle}{||\sin nx||^2} \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \; dx \\ &= \frac{2}{\pi} [-\frac{1}{n} x \cos nx|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \; dx] \\ &= \frac{2}{\pi} [-\frac{\pi}{n} (-1)^n + \frac{1}{n^2} \sin nx|_0^{\pi}] \\ &= \frac{2}{n} (-1)^{n+1} \end{aligned}$$

Question 2

Consider the piecewise smooth function f defind by

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ 2, & 0 \leq x < \pi \end{cases}$$

and

$$f(x + 2\pi) = f(x), \; x \in \mathbf{R}$$

(i) Sketch the function $f(x)$ on the interval $[-3\pi, 3\pi]$. What is the period of $f(x)$?

$$p = 2\pi$$

(ii) Find the Fourier series representation of $f(x)$ in $\mathcal{L}^2$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\; dx = \frac{1}{2\pi} \int_0^{\pi} 2\; dx = \frac{1}{2\pi} 2x|_0^{\pi} = \frac{1}{2\pi} 2\pi = 1$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\; dx = \frac{1}{\pi} \int_0^{\pi} 2 \cos nx\; dx = \frac{2}{n\pi} \sin nx|_0^{\pi} = 0$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\; dx = \frac{1}{\pi} \int_0^{\pi} 2 \sin nx\; dx = \frac{-2}{n\pi} \cos nx|_0^{\pi} = \frac{-2}{n\pi}((-1)^n - 1)$$

(iii) Find the sum $S_n(x)$ of the Fourier series at $x = 0$.

$$S_n(x) = 1 + \sum_{n=1}^{\infty}[\frac{2}{n\pi}(1-(-1)^n)\sin nx]$$
$$S_n(0) = 1 + \sum_{n=1}^{\infty}[\frac{2}{n\pi}(1-(-1)^n)\sin 0] = 1$$

OR

$$S_n(0) = \frac{1}{2}[f(0^+) + f(0^-)] = \frac{2}{2} = 1$$

(iv) Does the Fourier series converge point-wise to $f(\pi)$? Explain.

NO,

$$1 = S_n(\pi) \neq f(\pi) = 0$$

(v) Redefine the function f so that the Fourier series converge to f at every $x \in \mathbf{R}$.

$$f(x) = \begin{cases} 1 & x = -\pi \\ 0 & \pi \leq x \leq 0 \\ 1 & x = 0 \\ 2 & 0 \leq x \leq \pi \end{cases}$$

Question 3

Consider the eigenvalue problem

$$\begin{aligned} &Lu \ + \ \lambda u = 0, \qquad x \in [a,b] \\ &u(a) = 0, \qquad u(b) = 0. \end{aligned}$$

(i) Prove that if L is a self-adjoint operator, then $\lambda \in \mathbf{R}$.

Assume that L is a self-adj. operator $\implies \ -Lu = \lambda u$. For any $f \in \mathcal{L}^2$, we have

$$\begin{aligned} \lambda ||f||^2 &= \lambda \langle f, f \rangle \\ &= \langle \lambda f, f \rangle \\ &= \langle -Lf, f \rangle \\ &= \langle f, -Lf \rangle \\ &= \langle f, \lambda f \rangle \\ &= \bar{\lambda} \langle f, f \rangle \\ &= \bar{\lambda} ||f||^2 \end{aligned}$$

which implies that $\lambda = \bar{\lambda}$

(ii) Show that if

$$L = (1+4x^2)\frac{d^2}{dx^2} + 8x\frac{d}{dx}$$

in problem (1), then L is a self-adjoint operator.

$$\begin{aligned} &p(x) = 1 + 4x^2 \in C^2(\mathbf{R})\text{and real} \\ &q(x) = 8x \in C^1(\mathbf{R})\text{and real} \\ &r(x) = 0 \in C(\mathbf{R})\text{and real} \end{aligned}$$

which implies that it is formally self-adjoint. To prove that it is self adjoint, we have to prove

$$p(x)\big(f'\bar{g} - \bar{f}g'\big)_a^b = 0$$

but $u(a) = u(b) = 0$ on the boundary $\implies p(x)\big(f'\bar{g} - \bar{f}g'\big)_a^b = 0$

Question 4

I. Is the differential operator in the following boundary value problem self-adjoint? Justify your answer

$$\begin{aligned} &x^2u'' + xu' + \ \lambda u = 0, \qquad x \in [1, b] \\ &u(1) = 0, \qquad u(b) = 0. \end{aligned}$$

$$\begin{aligned} p(x) &= x^2 \in C^2(\mathbf{R})\text{and real} \\ q(x) &= x \in C^1(\mathbf{R})\text{and real} \\ r(x) &= 0 \in C(\mathbf{R})\text{and real} \end{aligned}$$

but $p' = 2x \neq q \implies L$ is not formally self-adjoint $\implies L$ is not self-adjoint.

II. Transform L into a self adjoint operator.

$$\rho(x) = \frac{1}{p(x)} e^{\int \frac{q}{p} \ dx} = \frac{1}{x^2} e^{\int \frac{x}{x^2} dx} = \frac{1}{x^2} e^{\ln x} = \frac{x}{x^2} = \frac{1}{x}$$

multiply the equation by ρ, we get

$$xu'' + u' + \frac{\lambda}{x} u = 0$$

III. Find the eigenvalues and eigenfunctions of the problem.

$$m(m-1)+m+\lambda = 0 \implies m^2+m-m+\lambda = 0 \implies m^2 = -\lambda \implies m = \pm i\sqrt{\lambda}$$

but $\lambda \geq \max|r(x)| = 0 \implies \lambda = 0$ or $\lambda > 0$

- when $\lambda = 0$ we have $u(x) = c_1 + c_2 x$

$$u(1) = c_1 + c_2 = 0$$

and

$$u(b) = c_1 + c_2 b = 0$$

$$\implies c_1 + c_2 = c_1 + bc_2 \implies c_2(b-1) = 0 \implies c_2 = 0$$

$$\mathbf{u(x) = c_1}$$

- when $\lambda > 0$ we have $u(x) = c_1 \cos\sqrt{\lambda}\ln x + c_2 \sin\sqrt{\lambda}\ln x$

$$u(1) = c_1 = 0$$

Using the B.C. we get,

$$u(1) = c_1 c_1 \cos\sqrt{\lambda}\ln(0) + c_2 \sin\sqrt{\lambda}\ln(0) = 0 \quad \implies c_1 = 0$$

$$u(b) = c_2 \sin\sqrt{\lambda}\ln b = 0 \implies \sin\sqrt{\lambda}\ln b = 0 \implies \sqrt{\lambda}\ln b = n\pi \implies \lambda = \frac{n^2\pi^2}{\ln^2 b},\ n \in \mathbf{N}$$

$$\mathbf{u(x) = c_2 \sin\frac{n\pi}{\ln b}x}$$