SOLVED SAMPLE

King Saud University
Department of Mathematics

First Midterm Exam

1^{st} **semester 1437 H**

Course Title: Math 425 (Partial Differential Equations)

Date: Nov 2016; (12-1:30)

(. . . . . .) Name . . . . ID . . . .

Question	Grade
Q1	5
Q2	10
Q3	5
Total	

Question 1

I) Classify each of the following PDEs as linear,quasilinear,or nonlinear and state its order and homogeneity .

(a) $x^2u_{xx} + y^2u_{yyy} - \log(1 + x^2)u = 0$

3^{rd} order, linear and homogeneous.

(b) $u_{tt} + uu_x + u_{xx} = f(x)$

2^{nd} order, quasilinear and non-homogeneous.

II) Find the general solution for

$$z_{xx} - 2z_{xy} + z_{yy} = 4e^{x+3y} + \cos(2x + y).$$

The differential operator

$$L = D_x^2 - 2D_xD_y + D_y^2$$

1. Solving $Lz = 0$:
   $Lz = (D_x - D_y)^2z = 0 \implies z_h = f(x + y) + xg(x + y)$
2. Particular solution:
   $z_{p_1} = \frac{4e^{x+3y}}{(1-3)(1-3)} = \frac{4}{(-2)(-2)}e^{x+3y} = e^{x+3y}$

   $z_{p_2} = \frac{\cos(2x+y)}{-1-2(-1)(2)-4} = -\cos(2x + y).$

The general solution is:

$$z = f(x + y) + xg(x + y) + e^{x+3y} - \cos(2x + y).$$

Question 2

I) Find the general solution of the following partial differential equations.

(i) $5p + 4q = y^3 + 1 + e^{2x}$ First order PDE with constant coefficients.

1. First we find the homogeneous solution Let $\xi = x$ and $\eta = 4x - 5y$, then

$$\begin{aligned} z_x = p &= z_\xi \xi_x + z_\eta \eta_x \\ &= z_\xi + 4z_\eta \\ z_y = q &= z_\xi \xi_y + z_\eta \eta_y \\ &= -5z_\eta \end{aligned}$$

Substitute in the PDE we get

$$5z_\xi + 20z_\eta - 20z_\eta = 0 \implies 5z_\xi = 0 \implies z_\xi = 0$$

By direct integration $z_h = f(\eta) = f(4x - 5y)$

2. For the particular solution

$$4z_y = y^3 + 1 \implies z_y = \frac{1}{4}(y^3 + 1) \implies z_{p_1} = \frac{1}{4}(\frac{1}{4}y^4 + y) = \frac{1}{16}y^4 + \frac{1}{4}.$$

$$5z_x = e^{2x} \implies z_x = \frac{1}{5}e^{2x} \implies z_{p_2} = \frac{1}{5}(\frac{1}{2})e^{2x} = \frac{1}{10}e^{2x}.$$

The general solution is

$$z = f(4x - 5y) + \frac{1}{16}y^4 + \frac{1}{4} + \frac{1}{10}e^{2x}.$$

(ii) $\cos y z_x + \cos x z_y = \cos x \cos y$
First order PDE with variable coefficients (linear)

Let $\xi = x$ and $\frac{dy}{dx} = \frac{\cos x}{\cos y} \implies \cos y dy = \cos x dx \implies \sin y = \sin x + c_1 \implies \eta = c_1 = \sin y - \sin x$. Then we find

$$\begin{aligned} z_x = p &= z_\xi \xi_x + z_\eta \eta_x \\ &= z_\xi - \cos x z_\eta \\ z_y = q &= z_\xi \xi_y + z_\eta \eta_y \\ &= \cos y z_\eta \end{aligned}$$

Substitute in the PDE we get

$$\begin{aligned} \cos y(z_\xi - \cos x z_\eta) + \cos x(\cos y z_\eta) &= \cos x \cos y \\ \cos y z_\xi - \cos y \cos x z_\eta + \cos y \cos x z_\eta &= \cos x \cos y \\ \cos y z_\xi &= \cos x \cos y \\ z_\xi &= \cos \xi \\ z &= \sin \xi + f(\eta) \end{aligned}$$

The general solution is

$$z = \sin x + f(\sin y - \sin x)$$

(iii) $xp + yq = 2xye^{-z}$ First order with variable coefficients (quasilinear). Using Lagrange method we have the auxiliary equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2xye^{-z}}$$

1. $\frac{dx}{x} = \frac{dy}{y} \implies \ln x = \ln y + c \implies x = e^c y \implies \frac{x}{y} = c_1.$
2. $\frac{ydx+xdy}{2xy} = \frac{dz}{2xye^{-z}} \implies ydx + xdy = e^z dz \implies xy = e^z + c_2 \implies c_2 = xy - e^z.$

The general solution is

$$z = F(\frac{x}{y},\ xy - e^z).$$

II) Show that $u(x, y) = \frac{1}{4}(x + f(x - y))^2$ is a solution of $u_x + u_y = \sqrt{u}$

$$u_x = \frac{1}{4}2(x + f(x - y))(1 + f') = \frac{1}{2}(x + f(x - y)) + \frac{1}{2}f'(x + f(x - y))$$

$$u_y = \frac{1}{4}2(x + f(x - y))f'(-1)$$

substitute in the PDE

$$\begin{aligned} u_x + u_y &= \frac{1}{2}(x + f(x - y)) + \frac{1}{2}f'(x + f(x - y)) - \frac{1}{2}f'(x + f(x - y)) \\ &= \frac{1}{2}(x + f(x - y)) \\ &= \sqrt{u} \end{aligned}$$

Question 3

Find the integral surface of

$$yp - 2xyq = 2xz,$$

which passes through the given curves $x = t,\ y = t^2\ and\ z = t^3$.

Using Lagrange method we have the auxiliary equation is

$$\frac{dx}{y} = \frac{dy}{-2xy} = \frac{dz}{2xz}$$

1. $\frac{dx}{y} = \frac{dy}{-2xy} \implies 2xdx = -dy \implies x^2 = -y + c_1 \implies c_1 = x^2 + y.$
2. $\frac{dy}{-2xy} = \frac{dz}{2xz} \implies \frac{dy}{-y} = \frac{dz}{z} \implies -\ln y + c = \ln z \implies c = \ln y + \ln z$
   $\implies e^c = c_2 = yz.$

The general solution is

$$F(x^2 + y,\ yz) = 0.$$

Using the conditions:

$$c_1 = t^2 + t^2 = 2t^2 \implies \frac{1}{2}c_1 = t^2 \implies \sqrt{\frac{1}{2}c_1} = t$$

$$c_2 = t^2t^3 = t^5 = (\sqrt{\frac{1}{2}c_1})^5 = \left((\frac{1}{2}c_1)^5\right)^{\frac{1}{2}}$$

$$\boxed{c_2^2 = \tfrac{1}{32}c_1^5}$$

Then

$$y^2z^2 = \frac{1}{32}(x^2 + y)^5.$$