1) $I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}$.

we know that:

$d\theta = \frac{dz}{iz}$ & $\sin\theta = \frac{1}{2i}(z - 1/z)$

then, $I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_{|z|=1} \frac{1}{2+\frac{1}{2i}(z-\frac{1}{z})} \frac{dz}{iz}$.

$$= \oint_{|z|=1} \frac{2i}{4i+z-\frac{1}{z}} \cdot \frac{dz}{iz}.$$

$$= 2\oint_{|z|=1} \frac{1}{z^2+4iz-1} dz$$

$$= 2\oint_{|z|=1} \frac{1}{(z-\underbrace{(-2i+\sqrt{3}i)}_{\text{inside}})(z-\underbrace{(-2i-\sqrt{3}i)}_{\text{outside}})} dz$$

$$= 4\pi i \, \text{Res}(f, -2i+\sqrt{3}i)$$

$$= 4\pi i \lim_{z\to z_0} (z - \underbrace{(-2i+\sqrt{3}i)}_{=z_0}) f(z)$$

$$= 4\pi i \frac{1}{-2i+\sqrt{3}i+2i+\sqrt{3}i}$$

$$= \frac{4\pi i}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}$$

3) : $I = \int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$

لأن الدالة المكاملة زوجية نعيد الكتابة :

$$I = \int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(3+2\cos\theta)^2}$$

$z = e^{i\theta}$, $\frac{dz}{iz} = d\theta$, $\cos\theta = \frac{z+\frac{1}{z}}{2}$.

$$= \frac{1}{2}\oint_{|z|=1} \frac{\frac{dz}{iz}}{\left(3+\cancel{2}\,\frac{z+\frac{1}{z}}{\cancel{2}}\right)^2}$$

$$= \frac{1}{2i}\oint_{|z|=1} \frac{\frac{dz}{z}}{\left(3+z+\frac{1}{z}\right)^2}$$

$$= \frac{1}{2i}\oint_{|z|=1} \frac{z}{(3z+z^2+1)^2}\,dz = \frac{1}{2i}\oint_{|z|=1} \frac{z\,dz}{(z-z_1)^2(z-z_2)^2}$$

ومن القانون العام

$$z_1 = \frac{-3+\sqrt{5}}{2} \in C \quad , \quad z_2 = \frac{-3-\sqrt{5}}{2} \notin C$$

$$I = \frac{1}{2i}\, 2\pi i \; Re(f; z_1)$$

$$\left(\frac{z}{z-z_2}\right)' = \frac{z-z_2-z}{(z-z_2)^2}$$

$$= \pi \lim_{z\to z_1} \frac{d}{dz}(z-z_1)^2 f(z)$$

$$= \pi \frac{-z_2}{(z_1-z_2)^2} = \frac{3\pi\sqrt{5}}{25} .$$

6-1:

1) b) $\dfrac{z+1}{z^2-3z+2}$

$z=1,\ z=2$ simple poles

$\text{Res}(2) = \lim_{z\to 2} (z-2)f(z) = 3$

$\text{Res}(1) = -2$

c) $\dfrac{\cos z}{z^2}$

$z=0$ pole of order 2

$\text{Res}(0) = 0.$

e) $\dfrac{e^z}{z(z+1)^3}$

$z=0,\ z=-1$

$z=0$: simple pole; $z=-1$: pole of order 3

$\therefore \text{Res}(0) = 1$

$$\text{Res}(-1) = \lim_{z\to -1} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z+1)^3 f(z)\right]$$

$$= \frac{1}{2} \lim_{z\to -1} \frac{d^2}{dz^2}\left(\frac{e^z}{z}\right)$$

$$= \frac{1}{2} \lim_{z\to -1} \frac{d}{dz}\left(\frac{z e^z - e^z}{z^2}\right)$$

$$= \frac{1}{2} \lim_{z\to -1} \frac{e^z\left[z^3 - 2z^2 + 2z\right]}{z^4}$$

$$= \frac{-5}{2} e^{-1}$$

g)

g) $\tan z = \dfrac{\sin z}{\cos z}$

$\cos z = 0 \longleftrightarrow \quad z_0 = k\pi + \frac{\pi}{2} \; ; k = 0, \pm 1, \pm 2, \dots$

all of them are simple poles.

From Example: if $f(z) = \dfrac{p(z)}{q(z)}$; $p(z), q(z)$ analytic at z_0, q has a simple zero at z_0, $p(z_0) \neq 0$, then:

$$\operatorname{Res}(f; z_0) = \frac{p(z_0)}{q'(z_0)}$$

$$\therefore \operatorname{Res}(f; z_0) = \frac{\sin z_0}{(\cos z)'\big|_{z=z_0}} = \frac{\sin z_0}{-\sin z_0} = -1$$

$$\therefore \operatorname{Res}\left(k\pi + \frac{\pi}{2}\right) = -1 \; ; k = 0, \pm 1, \pm 2, \dots$$