Exercises:

Let (N_t) be a Poisson r.v. with parameter λt that is.

$N_t \hookrightarrow \mathcal{P}(\lambda t)$.

Let $(U_k)_{k\geq 1}$ i.i.d.. $E[U_1] = \mu$ and $Var(U_1) = \sigma^2$.

N_t is $\perp\!\!\!\perp$ $(U_1, U_2, U_3, \ldots, U_n, \ldots)$.

$$R_t = U_1 + U_2 + U_3 + \cdots + U_{N_t} = \sum_{k=1}^{N_t} U_k.$$

Find $E[R_t | N_t]$? $Var(R_t | N_t)$? $\underbrace{E[X|Y]}_{= g(Y)}$?

Solution: $g(y) = E[X | Y = y]$; $y \in S_Y$.

First compute $E[R_t | N_t = n]$?

We have, $E[R_t | N_t = n] = E\left[\sum_{k=1}^{N_t} U_k \,\middle|\, N_t = n\right]$

$$= E\left[\sum_{k=1}^{n} U_k \,\middle|\, N_t = n\right]$$

$\boxed{E[E[Y|X]] = E[Y]}$

$$= \sum_{k=1}^{n} E[U_k | N_t = n]$$

$\boxed{E[E[X|Y]] = E[X]}$

$$= \sum_{k=1}^{n} E[U_k] \quad (U_k \perp\!\!\!\perp N_t \text{ for all } k \geq 1)$$

$$= \sum_{k=1}^{n} \mu = \mu n = g(n).$$

Then $E[R_t | N_t] = g(N_t) = \mu N_t$ ⊛

Deduce $E[R_t]$? Take the expectation in both sides of

⊛ $\underbrace{E[E[R_t | N_t]]}_{= E[R_t]} = E[\mu N_t] = \mu E[N_t]$

Therefore: $E[R_t] = \mu E[N_t] = \mu \lambda t$

$$Var(R_t \mid N_t = n) = E[R_t^2 \mid N_t = n] - \mu^2 n^2.$$

$$= E\left[\left(\sum_{k=1}^{N_t} U_k\right)^2 \Big| N_t = n\right] - \mu^2 n^2.$$

$$= \boxed{E\left[\left(\sum_{k=1}^{n} U_k\right)^2 \Big| N_t = n\right] - \mu^2 n^2}$$

We can also.

$$Var(R_t \mid N_t = n) = Var\left(\sum_{k=1}^{N_t} U_k \Big| N_t = n\right).$$

$$= Var\left(\sum_{k=1}^{n} U_k \Big| N_t = n\right)$$

$$E\left[\left(\sum_{k=1}^{n} U_k\right)^2\right] - \mu^2 n^2.$$

$$= E\left[\left(\sum_{k=1}^{n} U_k\right)^2\right] - \left(E\left(\sum_{k=1}^{n} U_k\right)\right)^2 = E X^2 - (E(X))^2$$

$$= Var\left(\sum_{k=1}^{n} U_k\right) = \sum_{k=1}^{n} Var(U_k) \quad (U_k)_{k\geq 1} \perp\!\!\!\perp.$$

$$= \sum_{k=1}^{n} \sigma^2 = n\sigma^2.$$

Then

$$Var(R_t \mid N_t) = N_t \sigma^2.$$

We need $Var(R_t)$?

$E[Var(X|Y)] +$

$+$ $Var(E[X|Y])$.

$Z = E[X|Y]$ r.v.

$E[Z]$, $Var(Z)$.

$R = Var(X|Y)$

$E[R]$ and $Var(R)$. ?

WA0P66

$$Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$$

① $$E[Var(X|Y)] = E[X^2] - E[(E[X|Y])^2]$$

$$Var(\underbrace{E[X|Y]}_{Z}) = E[(\underbrace{E[X|Y]}_{Z})^2] - (E[\underbrace{E[X|Y]}_{Z}])^2$$

$$Var(Z) = E[Z^2] - (E[Z])^2$$

② $$Var(E[X|Y]) = E[(E[X|Y])^2] - (E[X])^2$$

①+② $$\boxed{Var(X) = E[Var(X|Y)] + Var(E[X|Y])}$$

This is called the conditional variance relationship.

Application to $X = R_t$ and $Y = N_t$.

$$Var(R_t) = E[Var(R_t|N_t)] + Var(E[R_t|N_t])$$

$$= E[\sigma^2 N_t] + Var(\mu N_t)$$

$$= \sigma^2 E[N_t] + \mu^2 Var(N_t) = \sigma^2 \lambda t + \mu^2 \lambda t$$

$$= \lambda t(\sigma^2 + \mu^2).$$

$U_1 \hookrightarrow B(\frac{1}{2})$. Find $E[e^{zR_t}|N_t = n]$? $z \in \mathbb{R}$

$$E[e^{zR_t}|N_t = n] = E[e^{z\sum_{k=1}^{n} U_k} | N_t = n].$$

$$= E[e^{z\sum_{k=1}^{n} U_k}] \quad \text{because } (U_k) \perp\!\!\!\perp N_t \quad \sum_{k=1}^{n} U_k \perp\!\!\!\perp N_t$$

$$= E[e^{zU_1} e^{zU_2} \cdots e^{zU_n}]$$

$$= \underbrace{Ee^{zU_1}}\, \underbrace{Ee^{zU_2}} \cdots \underbrace{Ee^{zU_n}} = (E[e^{zU_1}])^n$$

$=$

W10P67

$$E\left[e^{zR_t} \mid N_t = n\right] = \left(M_{U_1}(z)\right)^n \quad \forall\, n \geq 0.$$

$$M_{U_1}(z) = E\left[e^{zU_1}\right] = e^{z\times 1}\frac{1}{2} + e^{z\times 0}\cdot\frac{1}{2} = \frac{e^z+1}{2}.$$

$$E\left[e^{zR_t} \mid N_t = n\right] = \left(\frac{e^z+1}{2}\right)^n. \quad z \text{ real number.}$$

Then:

$$E\left[e^{zR_t} \mid N_t\right] = \left(\frac{e^z+1}{2}\right)^{N_t}.$$

$$M_X(z) = E\left[e^{zX}\right], \quad z \in \mathbb{R}.$$

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

$$E\left[e^{zR_t}\right] = M_{R_t}(z) = E\left[E\left[e^{zR_t} \mid N_t\right]\right]$$

$$= E\left[\left(\frac{e^z+1}{2}\right)^{N_t}\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{e^z+1}{2}\right)^n e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$= e^{-\lambda t}\sum_{n=0}^{\infty}\left(\frac{\lambda t(e^z+1)}{2}\right)^n\cdot\frac{1}{n!}$$

$$= e^{-\lambda t}\, e^{\lambda t\left(\frac{e^z+1}{2}\right)}. \quad \blacksquare$$

W10P68

Discrete time Markov chains.

. Definitions and properties:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

__Def__: A sequence of r.v. $(X_n, n \geq 0)$ on a state space S. is a discrete-time Markov chain if:

- $\forall n \geq 0$, $X_n \in S$ that: $S_{X_n} = S$ $\forall n \geq 0$
- $\forall n \geq 1$ and for all $i_0, i_1, \dots, i_{n-1}, i_n \in S$, we have.

$$\mathbb{P}(X_n = i_n \mid X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

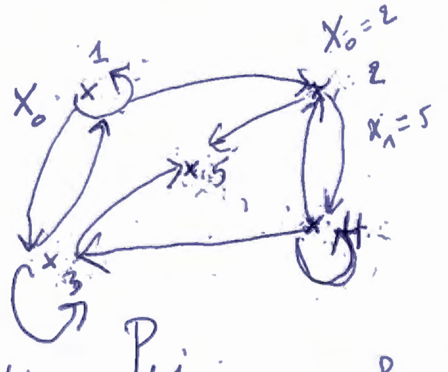
If $(X_n)_{n\geq 0}$ is a Markov chain; we denote by $\mathbb{P}(n,n+1)$ the matrix defined by $P_{ij}(n,n+1) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$.

$\mathbb{P}(n,n+1)$ is called the transition matrix of the M.C $(X_n)_{n\geq 0}$ called also the __one-step__ transition matrix.

__Def__: If $P(n,n+1)$ does not depend on n, (X_n) is called time homogeneous Markov chain: that is for all $i, j \in S$ we have: $\mathbb{P}(X_{n+k} = j \mid X_k = i) = \mathbb{P}(X_n = j \mid X_0 = i)$.

for all $n, k \in \mathbb{N}$.

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1/5 & 3/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



P_{21}

$P_{11} =$

$P_{55} = P(X_n = 5 \mid X_{n-1} = 5)$

$S = \{1, 2, 3, 4, 5\}$

$P_{5,4} = 0$

$P_{5,3} =$

We say that 5 is an absorbing state.

P_{ij} $\quad X_0 = 5$ $\quad P_{34}$

Example: $E = \{1, 2, 3\}$. $P(n,n+1) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \mathbb{P}$

Find $P_{22} = 0 = \mathbb{P}(X_{n+1} = 2 \mid X_n = 2)$.

$P_{ij} = P(X_{n+1} = j \mid X_n = i)$.

Compute: $\mathbb{P}(X_3 = 2 \mid X_4 = 1)$. $\boxed{P_{12} = P(X_{n+1} = 2 \mid X_n = 1)}$

$$P(X_3 = 2 \mid X_4 = 1) = P(X_3 = 2, X_4 = 1) / P(X_4 = 1).$$

$$= \frac{P(X_4 = 1 \mid X_3 = 2) P(X_3 = 2)}{P(X_4 = 1)} = P_{21}(3,4) \cdot \frac{P(X_3 = 2)}{P(X_4 = 1)}$$

let $\alpha_0 = (\alpha_0(i), i \in E)$ row vector be the initial distribution of X_0. that $\alpha_0(i) = \mathbb{P}(X_0 = i)$ for all $i \in E$.

$0 \leq \alpha_0(i) \leq 1$ and $\sum_{i \in E} \alpha_0(i) = 1$.

In general we denote by α_n the row vector giving the distribution of X_n for any $n \geq 0$: that

$$\alpha_n(i) = \mathbb{P}(X_n = i); \; i \in E.$$

Theorem: The stochastic process $(X_n)_{n \in \mathbb{N}}$ on the state space E is a Markov chain with initial distribution α_0 and transition matrix $\mathbb{P}(n,n+1)$ if and only if: for all $n \geq 1$, and for all $i_0, i_1, \ldots, i_n \in E$, we have:

$$\mathbb{P}(X_n = i_n, \ldots, X_0 = i_0) = \alpha_0(i_0) \prod_{k=1}^{n} P_{i_{k-1} i_k}(n,n+1).$$

$$= \alpha_0(i_0) P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$$

W10P70

$P(n,n+1)$ is called the one-step transition matrix.
More generally we denote $P_{ij}(m,n) = \mathbb{P}(X_n = j \mid X_m = i)$, the ~~probability that the~~ transition probability from state i at time m to state j at time n. $\boxed{n > m}$.

$\mathbb{P}(m,n) = \left(P_{ij}(m,n)\right)_{i,j \in E}$ is called the transition matrix fore m to n, also called $(n-m)$-step transition matrix:

<u>Proprerties:</u> ① $P(m,n) = P(m,m+1)P(m+1,m+2)\cdots\cdots \mathbb{P}(n-1,n)$

② $\mathbb{P}(m,n) = P(m,\ell)\mathbb{P}(\ell,n)$; $m < \ell < n$

① $\Rightarrow$ $P_{ij}(m,n) = \sum_{i_1 \in E} \cdots \sum_{i_{n-1} \in E} P_{i i_1}(m,m+1) P_{i_1 i_2}(m+1,m+2) \cdots\cdots P_{i_{n-1} j}(n-1,n)$.

$$\boxed{\alpha_n = \alpha_{n-1}\mathbb{P}(n-1,n)} \quad \forall n \geq 1.$$

In particular: $\alpha_1 = \alpha_0 \mathbb{P}(0,1)$, $\alpha_2 = \alpha_1 \mathbb{P}(1,2)$.

Hence $\boxed{\alpha_n = \alpha_0 P(0,1) P(1,2) \cdots\cdots \mathbb{P}(n-1,n)}$

If the M.C is <u>homogeneous</u> the $P(0,1) = P(1,2) = \cdots P(n-1,n)\ \forall n$
$= \mathbb{P}$.

Therefore: $\boxed{\alpha_n = \alpha_0 \mathbb{P}^n}$

W10P71

Let us go back to the example:

$$\alpha_3 = \alpha_2 P(2,3) = \alpha_1 P(1,2)\,P(2,3) = \alpha_0 P(0,1)\,P(1,2)\,P(2,3)$$
$$= \alpha_0 P^3 = \big(\alpha_3(1), \alpha_3(2), \alpha_3(3)\big).$$

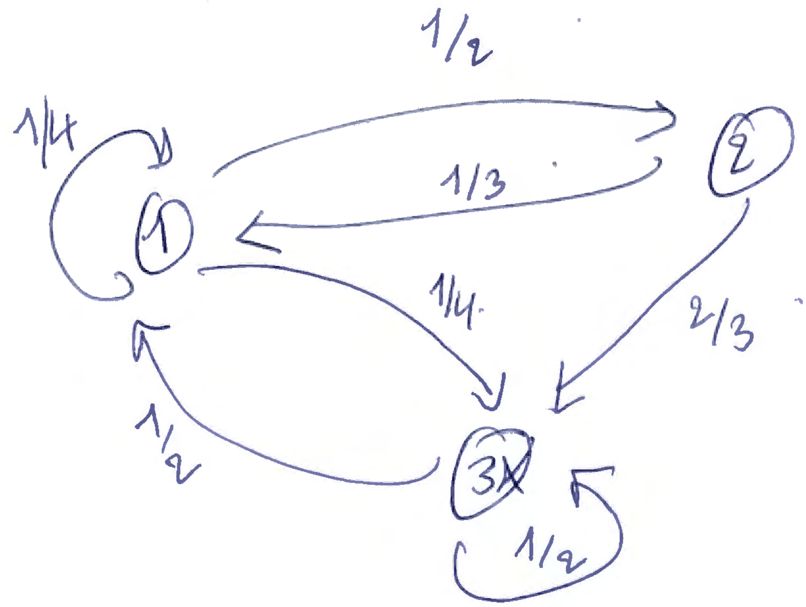
the same for α_4: (take $\alpha_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$).

$$\alpha_4 = \alpha_3 P.$$

$$P^3 = PPP = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\cancel{\big(\alpha_0(1), \alpha_0(2)}, \alpha_3(3)\big)\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = (\cdot, \cdot, \cdot).$$

Let us now plot the state diagram of the M.C.



This is called the state transition diagram.

W10P72