Examples. $E=\{1,2,3\}$.

① Let (X_n) be a M.C. with $P(6,7)=\begin{pmatrix}0.2 & 0.3 & 0.5\\ 0.3 & 0.5 & 0.2\\ 0.1 & 0.1 & 0.8\end{pmatrix}$

Find $P(X_7=j \mid X_6=1)_{j=1,2,3}$? $P(X_7=1 \mid X_6=i)$ $i=1,2,3$.

$$P(n,n+1)=\left(P_{ij}(n,n+1)\right)_{i,j\in E},\quad P_{ij}(n,n+1)=\mathbb{P}(X_{n+1}=j \mid X_n=i).$$

$P(X_7=1 \mid X_6=1)=0.2=P_{11}(6,7)$

$P(X_7=3 \mid X_6=1)=P_{13}(6,7)=0.5$.

$P(X_7=2 \mid X_6=3)=P_{32}(6,7)=0.1$.

$P(X_7=1 \mid X_6=3)=P_{31}(6,7)=0.1$.

② Find $\mathbb{P}(X_6=1 \mid X_7=3)=\dfrac{P(X_7=3;X_6=1)}{P(X_7=3)}$

$$=\frac{P(X_7=3 \mid X_6=1)\,P(X_6=1)}{P(X_7=3)}$$

$$=P_{13}(6,7)\,\frac{\alpha_6(1)}{\alpha_7(3)}.\quad \text{where } \alpha_6=\alpha_0 P(0,6).$$

If: $\alpha_6=\left(\frac{1}{2},0,\frac{1}{2}\right)$ then $\alpha_6(1)=\frac{1}{2}$.

$$\alpha_7=\alpha_6\mathbb{P}(6,7)=\left(\frac{1}{2},0,\frac{1}{2}\right)\begin{pmatrix}0.2 & 0.3 & 0.5\\ 0.3 & 0.5 & 0.2\\ 0.1 & 0.1 & 0.8\end{pmatrix}=\left(0.15,0.2,0.65\right)$$

So $\alpha_7(3)=0.65$. Hence $\mathbb{P}(X_6=1 \mid X_7=3)=\dfrac{0.5\times 0.5}{0.65}$.

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~~0.1+0.05~~

Find $\mathbb{P}(X_7 = 2 \mid X_6 = 3, X_5 = 1, X_3 = 2)$.

$= P(X_7 = 2 \mid X_6 = 3)$ by definition of the M.C.

$= P_{32}(6,7)$

Find ($P(X_7 = 2 \mid X_5 = 3, X_4 = 1, X_4 = 2)$.

$= P(X_7 = 2 \mid X_5 = 3) = P_{32}(5,7)$.

To find $P_{32}(5,7)$, we first calculate $P(5,7)$ ~~[illegible]~~

which is given by $P(5,6) \times \mathbb{P}(6,7)$.

In general if we are asked to compute

$P(X_{n+1} = j \mid X_{n-1} = i)$ we need a ~~two~~-step

transition probability matrix $\mathbb{P}(n-1, n+1)$.

$$P_{ij}(n-1,n+1) = \sum_{k \in E} P_{ik}(n-1,n)\, P_{kj}(n,n+1).$$

We have $P_{ij}(n-1,n+1) = \mathbb{P}(X_{n+1} = j \mid X_{n-1} = i)$

$= \mathbb{P}(X_{n+1} = j, X_n \in E \mid X_{n-1} = i)$

$= \sum_{k \in E} \mathbb{P}(X_{n+1} = j, X_n = k \mid X_{n-1} = i)$.

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$$P_{ij}(n-1,n+1) = \sum_{k\in E} \frac{P(X_{n+1}=j,\ X_n=k,\ X_{n-1}=i)}{P(X_{n-1}=i)}.$$

$$= \sum_{k\in E} \frac{P(X_{n+1}=j \mid X_n=k, X_{n-1}=i)\, P(X_n=k, X_{n-1}=i)}{P(X_{n-1}=i)}.$$

$$= \sum_{k\in E} P_{kj}(n,n+1)\, P(X_n=k \mid X_{n-1}=i).$$

$$= \sum_{k\in E} P_{ik}(n-1,n)\, P_{kj}(n,n+1).$$

If the M.C is homogeneous we get

$$P^{(2)}_{ij} = \sum_{k\in E} P_{ik} P_{kj} = \left(P^2\right)_{ij}.$$

<u>Example</u>: Find $P(X_2=1 \mid X_4=2)$

$E=\{1,2,3\}$; $P(0,1) = P(n,n+1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}.$

$$P(X_2=1 \mid X_4=2) = \frac{P(X_4=2, X_2=1)}{\alpha_4(2)} = \frac{P(X_4=2 \mid X_2=1)\, P(X_2=1)}{\alpha_4(2)}$$

$$= P_{12}(2,4)\,\frac{\alpha_2(1)}{\alpha_4(2)} = P^{(2)}_{12}\,\frac{\alpha_2(1)}{\alpha_4(2)}.$$

Where $P_{12}(2,4) = P^2_{12} = \sum_{k=1}^{3} P_{1k}P_{k2} = \text{line}(1)\cdot\text{column}(2).$

$$\alpha_4 = \alpha_3 P = \alpha_2 P^2 = \alpha_1 P^3 = \alpha_0 P^4.$$

W11 P75 $\quad \alpha_2 = \alpha_0 P^2 \quad$ take $\alpha_0 = (0.2, 0, 0.8) = \text{dist}(X_0)$.

or $\alpha = (1,0,0)$, $X_0 = 1$.

Classification of states.

Def: "Accessibility": We say that state j is accessible from i, written as $i \longrightarrow j$ If $P_{ij}^{(n)} > 0$ for some n.

where $P^{(n)}$ is the n step transition probability matrix.

that is $P_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i)$.

We assume every state is accessible from itself since:

$$P_{ii}^{(0)} = P(X_0 = i \mid X_0 = i) = 1 \quad \forall i \in E.$$

Def: Two state i and j are said to communicate, written $i \longleftrightarrow j$ if they are accessible from each other:

$$\boxed{i \longleftrightarrow j \iff i \longrightarrow j \text{ and } j \longrightarrow i}$$

Therefore, the stats of a Markov Chain can be partioned into communicating classes such that only members of the same class communicate with each other. That is two stats i and j belong to the same class if and only if $i \longleftrightarrow j$.

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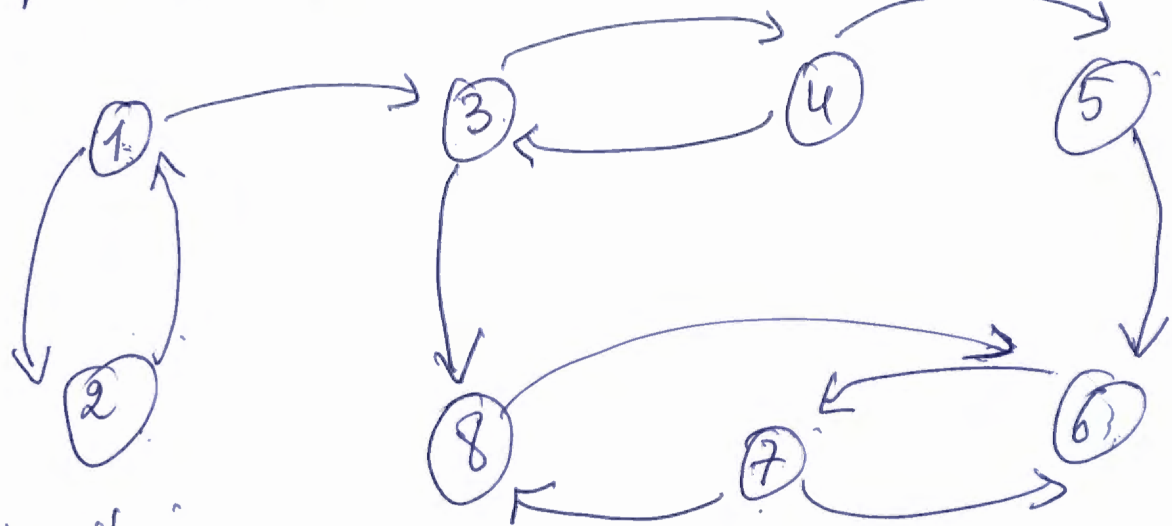
Examples: $E = \{1, 2, \dots, 8\}$:



transition states Diagram of a M.C.

Find the communicating classes.

$1 \longrightarrow 2 \longrightarrow 1$; $\quad 6 \longrightarrow 7 \longrightarrow 8 \longrightarrow 6$.

$3 \longrightarrow 4 \longrightarrow 3$

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class 1 $= \{1, 2\}$, class 2 $= \{3, 4\}$, class 3 $= \{6, 7, 8\}$

class 4 $= \{5\}$ $\square$

Def: A Markov chain is said to be irreducible if it has only one communicating class.

Irreducibility is a desirable property in the sense that it can simplify analysis of the limiting behavior.

A M.C. is said to be irreducible if all states communicate with each other.

W11 P77.

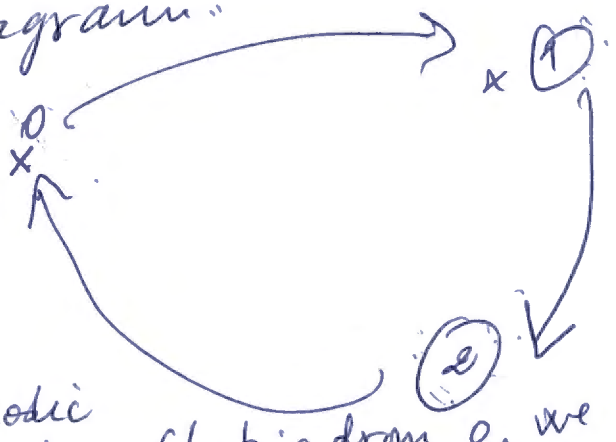
In the previous example if at any time the M.C. enters class 3, it will always stay in that class. On the other hand, for other classes this is not true.
For example if "$X_0 = 1$" that is the M.C. starts from state 1, then the M.C. might stay in class 1 for a while, but at some point it will leave that class and it will never return to that class again.

Def. The states in class 3 are called <u>recurrent</u> states, while the other states in the chain are called <u>transient</u>.

For any state i, we define $f_{ii} = P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i)$.

State i is recurrent if $f_{ii} = 1$ and it is transient if $f_{ii} < 1$.

<u>Periodicity</u>: consider the M.C. given by its state transition diagram:



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$p_{00}^{(7)} = 0$

$p_{00}^{(8)} = 0$,

There is a periodic pattern in this chain. Starting from 0, we only return to 0 at times 3, 6, 9, ... In other words

$p_{00}^{(n)} = 0$, if n is not divisible by 3.

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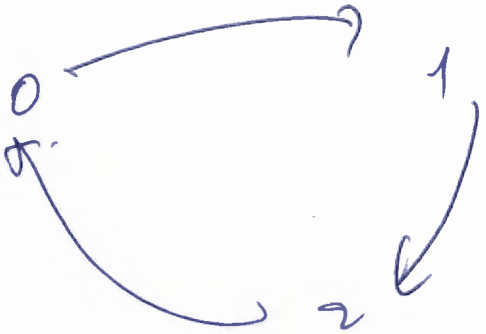
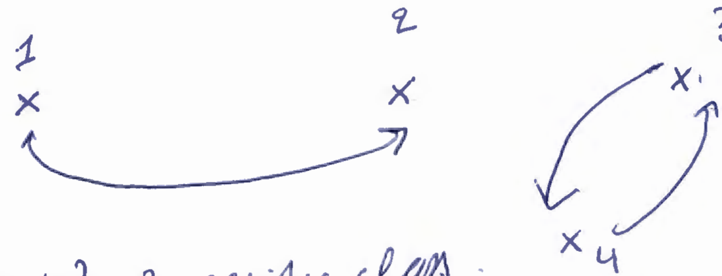


State 0 is called a periodic state with priod $d(0) = 3$.
also $d(1) = 3$ and $d(2) = 3$.

The period of a state i is the largest integer d satisfying the following property: $p_{ii}^{(n)} = 0$ whenever n is not divisible by d.
The period of i is shown by $d(i)$.
If $p_{ii}^{(n)} = 0$ for all $n > 0$, then we set $d(i) = +\infty$.
If $d(i) > 1$ we say that state i is periodic.
If $d(i) = 1$ " " " " " is aperiodic.

$\boxed{\text{M.C.}}$

Example:



class 1 = {3, 4} 2-periodic class.
class 2 = {1} aperiodic.
class 3 = {2} aperiodic.
This M.C. is not irreducible (is reducible) because it has more than one class.
consider a M.C with state transition diagram.

$$\begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

This M.C is irreducible.
$1 \to 2 \to 4 \to 3 \to 1$
This is 4 periodic class.

W11P79 :