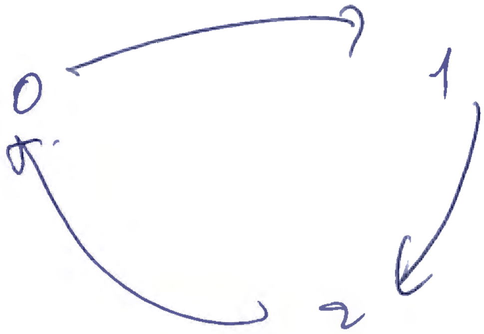


State 0 is called a periodic state with priod $d(0) = 3$.

also $d(1) = 3$ and $d(2) = 3$.

The period of a state i is the largest integer d satisfying the following property: $p_{ii}^{(n)} = 0$ whenever n is not divisible by d.

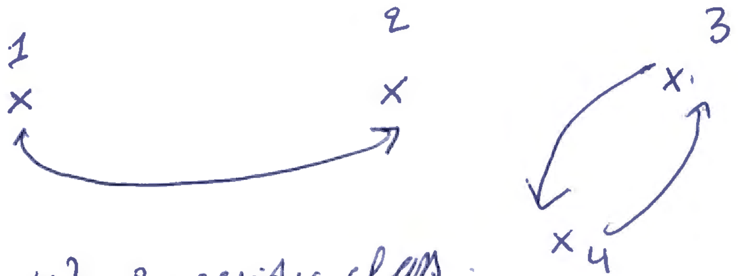
The period of i is shown by $d(i)$.

If $p_{ii}^{(n)} = 0$ for all $n > 0$, then we det $d(i) = +\infty$.

If $d(i) > 1$ we say that state i is periodic.

If $d(i) = 1$ " " " " " is aperiodic.

Example: 3. $\boxed{\text{M.C.}}$



class 1 = $\{3, 4\}$ 2-periodic class.

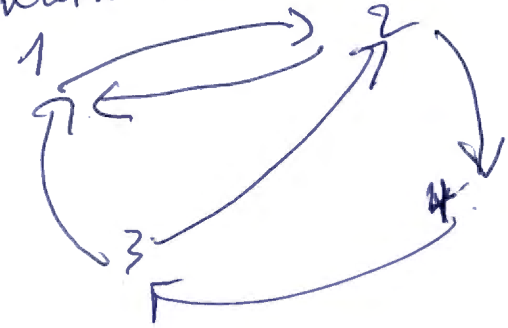
class 2 = $\{1\}$ aperiodic.

class 3 = $\{2\}$ aperiodic.

This M.C. is not irreducible (is reducible) because it has more than one class.

Consider a M.C with state transition diagram.

$$\begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$



This M.C is irreducible.

$1 \to 2 \to 4 \to 3 \to 1$

This is 4 periodic class.

W11P79:

$$P(X_3=1/X_5=1) = \frac{P(X_5=1,\ X_3=1)}{P(X_5=1)}$$

Exercises

$$= \frac{P(X_5=1|X_3=1)\,P(X_3=1)}{P(X_5=1)}$$

$$= \frac{P_{11}(3,5)\,\alpha_3(1)}{\alpha_5(1)}.$$

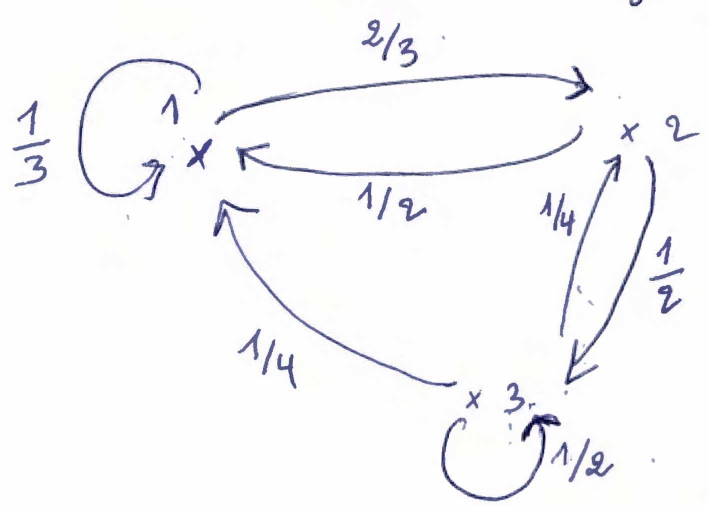
If we are given α_3, $\alpha_5 = \alpha_3 P(3,4)P(4,5)$.

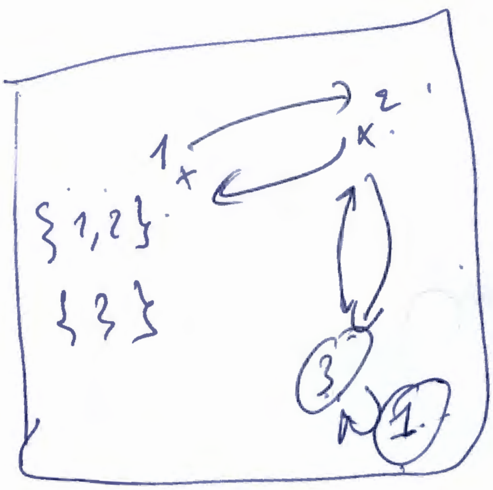
$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$E = \{1,2,3\}$.
transition matrix of an homogeneous M.C.

- Find communicating classes of this M.C.

① Draw the state transition diagram.





$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$, there is one communicating class

$C = E \Rightarrow$ recurrent because each state can be visited infinitely many times.

The M.C is irreducible because it has only one communicating class.

$\frac{2}{3} = \frac{4}{6}$

The class E is aperiodic.

W12 P80

A review on exponential distribution.

let X be a r.v, X is said to have an exponential distribution with parameter λ, $\lambda>0$; if its probability density function "pdf" is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & < 0 \end{cases}$$

or equivalently if its c.d.f. is given by:

$$F_X(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \quad ; \quad Var(X) = \frac{1}{\lambda^2}.$$

Moreover the M.G.F $M_X(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}, \ t<\lambda$

Remember that we can compute all moments of X by differentiating the $M_X(t)$ at time 0.

for example: $E[X^2] = M_X''(0)$.

$$M_X'(t) = \frac{\lambda}{(\lambda-t)^2} \quad \text{and} \quad M_X''(t) = \frac{+2(\lambda-t)\lambda}{(\lambda-t)^4} = \frac{2\lambda}{(\lambda-t)^3}$$

$$M_X''(0) = \frac{2}{\lambda^2}.$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

Example: (Exponential r.v. and expected discounted returns). Suppose that you are receiving rewards at randomly changing rates continuously throughout time. Let $R(t)$ denote the random rate at which you are receiving rewards at time t.

W12 P80

For a value $\alpha \geq 0$, called the discount rate, the quantity

$$R = \int_0^{+\infty} e^{-\alpha t} R(t)\, dt$$ represents the total discounted reward.

(in some aplications α is a continuously compounded interest rate, and R is the present value of the infinite flow of rewards.

Since R is a r.v. we are interested to compute its expectation.

$$E[R] = \int_0^{+\infty} e^{-\alpha t} E[R(t)]\, dt.$$

Let T be an exponential r.v. with rate (parameter α) that is independ of all the r.v. $R(t)$.

$$E[R] = \int_0^{+\infty} e^{-\alpha t} E[R(t)]\, dt = E\left[\int_0^T R(t)\, dt\right].$$

Let us denote by $I(t) = \begin{cases} 1 & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases} := I(t).$

$$\int_0^T R(t)\, dt = \int_0^{+\infty} R(t) I(t)\, dt.$$

$$\cancel{E[R] = \int_0^{+\infty} e^{-\alpha t}}$$

$$E\left[\int_0^T R(t)\, dt\right] = E \int_0^{\infty} R(t) I(t)\, dt = \int_0^{\infty} E[R(t) I(t)]\, dt$$

$$= \int_0^{+\infty} E[R(t)]\, E[I(t)]\, dt \qquad R(t) \perp\!\!\!\perp I(t)$$

$$= \int_0^{+\infty} E[R(t)]\, P(T \geq t)\, dt$$

$$= \int_0^{+\infty} e^{-\alpha t} E[R(t)]\, dt. \square$$

W19P81

Properties of the exponential distribution.

Def: A r.v. X is said to be without memory or memoryless, if

$$\circledast \quad P(X > s+t \mid X > t) = P(X > s) \text{ for all } s,t \geq 0$$

If we think of X as being the lifetime of some instrument then equation $\circledast$ states that the probability that the instrument lives for at least $s+t$ hours given that it has survived t hours is the same as the initial probability that it lives for at least s hours.

The equation $\circledast$ is equivalent to:

$$P(X > s+t) = P(X>s)P(X>t).$$

Example: If X is exponentially distributed then

$$P(X > s+t) = e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$$

$$= P(X>s)P(X>t).$$

$\boxed{F_X(t) = 1 - \cancel{\lambda} e^{-\lambda t} \text{ c.d.f}}$

It follows that exponential distributions are memoryless.

Example: The dollar ~~amont~~ amount of damage involved in an automobile accident is an exponential random variable with mean 1000.

For this policy the insurance company only pays that amount exceeding (the deductible amount of)400.

• Find the expected value and the standard deviation of the amount the insurance company pays per accident.

Solution: If X is the dollar amont of damage resulting from an accident, then the amount paid by the insurance company is $(X-400)^+ = \max(X-400, 0)$.

We shall compute: $E[\max(X-400,0)]$ and $Var(\max(X-400,0))$.

W12 P8 2

$$E[(X-400)^+] = \int_{400}^{+\infty} (x-400)\frac{1}{1000}e^{-\frac{x}{1000}}\,dx\ :$$

the same for the variance : ($\lambda = \frac{1}{1000}$).

$$E[((X-400)^+)^2] = \int_{400}^{+\infty} (x-400)^2 \lambda e^{-\lambda x}\,dx.$$

~~If X is a memoryless r.v.~~

$X \hookrightarrow \exp(\lambda)$.

$$\begin{cases} E[(X-t)^+] = \dfrac{e^{-\lambda t}}{\lambda} \\ E[\min(X,t)] = \dfrac{1}{\lambda} - \dfrac{1}{\lambda}e^{-\lambda t} \end{cases}$$

Def Poisson process

counting processes..

Def: A stochastic process $(N_t, t \geq 0)$ is said to be a counting process if N_t represents the total number of "events" that occur by time t.

Examples ① If we let N_t equals the number of persons who enter a particular store at or prior to time t. Then $N_t, t \geq 0$ is a counting process in which an event corresponds to a person entering the store.

② If we say that an event occurs whenever a child is born, then $N_t, t \geq 0$ is a counting process equals the total number of people who were born by time t.

W12P93

② ~~The~~ From its definition we see that a counting process N_t satisfy the following properties.

① $N_t \geq 0$ for all $t \geq 0$

② N_t is interger valued.

③ if $s < t$; then $N_s \leq N_t$

④ For all $s < t$, $N_t - N_s$ equals the number of events that occur in the interval $(s, t]$.

Def: A counting process is said to ~~be~~ possess independent increments if the numbers of events that occur in disjoint time intervals are independent.

Example: N_{10} must be independent of $N_{15} - N_{10}$

Def: The counting process $(N_t, t \geq 0)$ is said to be a Poisson process having rate $\lambda > 0$, if:

(i) $N_0 = 0$

(ii) $\forall t_0 < t_1 < t_2 < \dots < t_n$ the random variables $N_{t_n} - N_{t_{n-1}}, N_{t_{n-1}} - N_{t_{n-2}}, \dots, N_{t_2} - N_{t_1}, N_{t_1} - N_{t_0}$ are independent.

(iii) for all $s, t \geq 0$ ~~Nt~~ $N_{t+s} - N_s \hookrightarrow \mathcal{P}(\lambda t)$.

W12 P84.

That is $\mathbb{P}(N_{t+s} - N_s = n) = e^{-\lambda t} \dfrac{(\lambda t)^n}{n!}$.

$= \mathbb{P}(N_t = n)$.

$E[N_t] = \lambda t$ and $Var(N_t) = \lambda t$.

Do it as exercise.

<u>Properties:</u> Let $(N_t)_{t \geq 0}$ be a Poisson process having rate $\lambda > 0$:

$$P(N_h = 1) = \lambda h + o(h).$$

and $P(N_h \geq 2) = o(h)$:

where ~~lim~~ $o(h)$ is a function of h such that $\lim\limits_{h \to 0} \dfrac{o(h)}{h} = 0$.

From the previous properties we can show that: $P(N_h = 0) = 1 - \lambda h + o(h)$.

and $M_{N_t}(u) = e^{\lambda t(e^u - 1)}$. {moment generating function of N_t}.

W12P85