Introduction :

If X is a r.v. let g be a given function.

$g(X)$ is also a r.v.

$E[g(X)]$ (compution of the expectation of $g(X)$) ..

We need the distibution of X.

1) Discrete case : i) {all possible values of X} $= S$
sample space.
ii) probabilities of corresponding to possible values :

Example: i) $S = \{x_1, x_2, x_3 \ ---\}$.

ii) $p_i = P(X = x_i)$.

$(x_i, p_i)_{i \in I} \longrightarrow$ distubution of X.

In this case: $E[g(X)] = \sum_i p_i\, g(x_i)$.

$E[X] = \sum_i p_i x_i \quad | \quad g(x) = x$.

$E[X^2] = \sum_i p_i x_i^2$;

$n \geq 2 \quad E[X^n] = \sum_i p_i x_i^n$

W2-P1

~~Case~~
2) Continuous case: let X be a.r.v. the distribution of X is given by the c.d.f $F_X(x) = P(X \le x)$ for all x in $S = \{$set of all possible values of $X\}$.
or by the p.d.f. $f_X(x) = F_X'(x)$.

$$E[g(X)] = \int_S g(x) f_X(x)\,dx$$

change of v. formula
integration by parts
direct calculations.

If: $g(x) = x \longrightarrow E[g(X)] = E[X]$.
If $g(x) = x^2 \longrightarrow E[g(X)] = E[X^2]$.
Hence we can compute the variance.

To compute $E[g(X)]$ we need the distribution of X.

Example ① consider a cube numbered from 1 to 6.
If X stands for the ~~sum~~ number of the upper face.
Find the distribution of X:

(i) $S = \{1, 2, \dots, 6\}$. some times denoted by $f(k)$
(ii) for any $k \in S$, $P_k = P(X = k) = f(k) = \frac{1}{6}$.

Example: ② let X be a r.v. taking values in $\{5, 13, 9, 10\}$ (x_1, x_2, x_3, x_4).

$P_1 = P(X=5) = \frac{1}{4}$, $P_2 = P(X=13) = \frac{1}{5}$; $P(X=9) = \frac{1}{8}$.

$P(X=10) = 1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{8}$. $(x_i, P_i)_{i=1,\dots,4} \longrightarrow$ distribution

W2-P2

Exercises:

Ex(1). consider $A = \{a, b, c\}$:

Q.: Find the power set of A: $\mathcal{P}(A)$:

A: $\mathcal{P}(A)$: is the set of all subsets of A.

then $\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

$\#\mathcal{P}(A) = 8 = 2^3$. In general if $\#A = n$

then $\#\mathcal{P}(A) = 2^n$.

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

~~Ex(2): Find $\sum_{k=1}^{n} (2k-1)$.~~

$A = \{x_1, x_2, \dots, x_n\}$ for all $i \neq j$, $x_i \neq x_j$

$$\#\mathcal{P}(A) = C_n^0 + C_n^1 + C_n^2 + C_n^3 + \dots + C_n^n$$

$$= \sum_{k=0}^{n} C_n^k 1^k 1^{n-k} = (1+1)^n = 2^n.$$

Thanks to the binomial formula:

$$\sum_{k=0}^{n} C_n^k x^k y^{n-k} = (x+y)^n \quad \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}.$$

Ex(2). $$\sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - n$$

$$= 2\,\frac{(1+n)n}{2} - n = n^2.$$

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}; \quad \sum_{k=1}^{n} a\,\frac{1-a^n}{1-a}.$$

$$\sum_{k=m}^{n} a^k = a^m\,\frac{1-a^{n-m+1}}{1-a}.$$

~~W2-2-P1~~

$$\lim_{n\to+\infty} \sum_{k=0}^{n} a^k = \lim_{n\to+\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a} - \frac{1}{1-a} \lim_{n\to+\infty} a^{n+1}$$

We just need to calculate $\lim_{n\to+\infty} a^{n+1}$?

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & |a|<1 \\ +\infty & a>1 \\ 1 & a=1 \\ \text{not exist} & a=-1 \\ \text{not exists} & a<-1 \end{cases}$$

$(-1)^n$ does not have a limit.

$(-2)^n$ $\nexists$.

$$\boxed{\lim_{n\to+\infty} \sum_{k=0}^{n} a^k = \frac{1}{1-a} \quad \text{for } \forall\, |a|<1.}$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad \forall\, |a|<1.$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Find: $\displaystyle\sum_{k=1}^{n} k^2 = 1 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Ex 3:

N balls

1, 2, ... , N

urn

let X be the number of the extracted ball. $X: \Omega \longrightarrow S$, $\omega \longmapsto X(\omega)$.

Find the distribution of X.

X is a discrete r.v.: i) $S = X(\Omega) = \{1, 2, \dots, N\}$

ii) P_i, $i \in S$: $P_i = P(X=i) = \frac{1}{N}$

This a uniform discrete distribution.

$$E[X] = \sum_{i=1}^{N} p_i i = \frac{1}{N} \sum_{i=1}^{N} i = \frac{1+N}{2}.$$

$$E[X^2] = \sum_{i=1}^{N} p_i i^2 = \frac{1}{N} \sum_{i=1}^{N} i^2 = \frac{(N+1)(2N+1)}{6}.$$

$$Var(X) = E[X^2] - (E[X])^2 = E\left[(X - E[X])^2\right] \geq 0$$

$$Sd(X) = \sqrt{Var(X)}.$$

<u>consequently</u>: $(E[X])^2 \leq E[X^2]$

<u>Proprerties of the expectation</u>:

Let $X_1, X_2, X_3, \ldots, X_n$ be r.v and $(\alpha_i)_i \in \mathbb{R}$.

$$E\left[\sum_{i=1}^{n} \alpha_i X_i\right] = \sum_{i=1}^{n} \alpha_i E[X_i].$$

in particular: $E[X_1 + X_2] = E[X_1] + E[X_2]$

and $E[\alpha X] = \alpha E[X]$.

(i) $Var(\alpha X) = \alpha^2 Var(X). \quad \forall \alpha \in \mathbb{R}$

(ii) $Var(X + a) = Var(X) \quad \forall a \in \mathbb{R}.$

<u>Homework</u> prove (i) and (ii).

Homework:

i/- Prove that $Var(\alpha X) = \alpha^2 Var(X)$

ii/- and $Var(a+X) = Var(X)$..

iii/ $E[aX+b] = aE[X]+b$.

iv/ Give an example of a uniform distribution on finite sample space.

Example: $S = \{4, 6, 8\}$, $P_1 = \frac{1}{3} = P_2 = P_3$:

~~Distribution~~ Uniform distribution with mean 0.

$S = \{-4; -3; 3; 4\}$. $P_i = \frac{1}{4}$.

$S = \{-4, 4\}$; $P_i = \frac{1}{2}$

It is possible to have a random variable with variance 0? $S = \{4\}$

If X is a r.v such that $Var(X) = 0$

then $X = \text{constant} = E[X]$.

There is no randomness.

~~$E[e^X] = \sum_{k=1}^{4} p_k e^{x_k} = \frac{1}{4}e^{5} + \frac{1}{5}e^{13} + \frac{1}{8}e^{29} + \frac{11}{40}e^{10}$~~

~~in this case $g(x) = e^x$~~

Definition: The sample space S for a probability experiment is the set of all possible outcoms of the experiment.

Examples: (1) A single die is rolled and the number facing up recorded. The sample space $S = \{1, 2, \ldots, 6\}$.

(2) A coin is tossed and the side facing up is recorded. The sample space is $S = \{H, T\}$.

(3) (Death of an insured). An insurance company is interested in the probability that an insured will die in the next year. The sample space S:

$S = \{\text{death, survival}\}$.

(4) (Default of a bond): Companies borrow money they need by issuing bonds. A bond is typically sold in \$1000 units which have a fixed interest rate such as 8% per year for twenty years. When you buy a bond for \$1000, you are actually loaning the company your \$1000 in return for 8% interest per year. You are supposed to get your \$1000 loan back in twenty years.

The sample space $S = \{\text{default, no default}\}$.

Example: An insurance company has sold 100 individual life insurance policies.
When an insured individual dies the beneficiary named in the policy will file a claim for the ~~amount~~ amount of the policy.
You wish to observe the number of claims filed in the next year. The sample space consists of all integers from 0 to 100, so $S = \{0, 1, 2, \dots, 100\}$.

Let S be a finite sample space and let $\mathcal{P}(S)$ be its power set. $\#\mathcal{P}(S) = 2^{\#S}$.
Denote by $n = \#S$. then $\#\mathcal{P}(S) = 2^n$.
~~Consider~~ ~~an~~ all the element of $\mathcal{P}(S)$ are called events.
Let $A \in \mathcal{P}(S)$ that is A is a part of S.
A^c: the complementary of A; $A^c \in \mathcal{P}(S)$.
If $A \in \mathcal{P}(S)$ and $B \in \mathcal{P}(S)$ then:
$A \cap B \in \mathcal{P}(S)$ and $A \cup B \in \mathcal{P}(S)$.
In general if $(A_i)_{i=1}^{n}$ is a familly of events in $\mathcal{P}(S)$.
then $\bigcup_{i=1}^{n} A_i \in \mathcal{P}(S)$ and $\bigcap_{i=1}^{n} A_i \in \mathcal{P}(S)$.
• $\phi \in \mathcal{P}(S)$; and $S \in \mathcal{P}(S)$.

~~W2-P4~~

Let us now make a review on very well known distributions:

i/ sample space.

ii/ $\begin{cases} \text{p.m.f : probability mass function (for discrete distributions)} \\ \text{p.d.f or c.d.f : (for continuous distributions).} \end{cases}$

Examples ① Bernoulli distribution: $B(p)$:

- $S = \{\text{true, false}\}$ or $\{0, 1\}$:
- p.m.f : $(0, 1-p)$ and $(1, p)$. where $0<p<1$

② Binomiale distribution $B(n,p)$. $0<p<1, 1\leq n$

ⓘ $S = \{0, 1, \dots, n\}$, ⓘⓘ (k, p_k) p.m.f. $k=1, \dots n$

where $p_k = C_n^k p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$.

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+1-p)^n = 1.$$

③ Geometric distribution: $G(p)$.

i) $S = \{1, 2, \dots\} = \mathbb{N}^*$

ii) $\forall k \geq 1$, $p_k = (1-p)^{k-1} p$.

$$\sum_{k=1}^{\infty} p_k = p \sum_{k=1}^{\infty} (1-p)^{k-1} = p \sum_{j=0}^{\infty} (1-p)^j$$

$$= p \left(\frac{1}{1-(1-p)} \right) = \frac{p}{p} = 1.$$

Homework calculate $E[G(p)] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$.