Axioms of Probability theory:

let Ω be an nonempty set : $\Omega \neq \Phi$.

Def. A collection $\mathcal{F}$ of subsets of Ω, $(\mathcal{F} \subseteq \mathcal{P}(\Omega))$ is called an algebra if:

i) $\Phi, \Omega \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

iii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

$\boxed{\text{Def.}}$ A collection $\mathcal{F}$ of subsets of Ω is called a σ-algebra if:
$\mathcal{F}$ is an algebra and for any familly $(A_n)_{n \geq 0}$ in $\mathcal{F}$:
we have $\bigcup_{n \geq 0} A_n \in \mathcal{F}$.

A sigma-algebre (σ-algebra) is some times called σ-field.

Examples: $\mathcal{F} = \{\Phi, \Omega\}$ is σ-algebra

If $A \subset \Omega$: $\mathcal{F} = \{\Phi, A, A^c, \Omega\}$ is a σ-algebra.

$\mathcal{F} = \mathcal{P}(\Omega)$ is an algebra (finite) and is σ-algebra (in general)

Déf: A probability measure $\mathbb{P}$ is a mapping:

$$\mathbb{P} : \mathcal{F} \longmapsto [0,1] \quad \Big| \text{ satisfying:}$$
$$A \longmapsto \mathbb{P}(A).$$

(i) $\mathbb{P}(\Omega) = 1$.

(ii) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \Big/ A, B \in \mathcal{F}$.

(iii) For any finite collection of pairwise disjoint sets $A_1, A_2, \dots, A_n \in \mathcal{F}$: $(A_i \cap A_j = \phi \ \forall i \neq j)$.

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{P}(A_k).$$

Properties of a probability measure:

- $\mathbb{P}(\Phi) = 0$; $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
- If $A \subseteq B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$.

. $(\Omega, \mathcal{F})$ is called a measurable space.

• $(\Omega, \mathcal{F}, \mathbb{P})$ is called a ~~measeb~~ measured space or a probability space.

Ω → nonempty set

$\mathcal{F}$ → σ-algebra

$\mathbb{P}$ → a probability measure.

Déf: Random variables:

A real random variable is a mapping X from: $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algerabra on $\mathbb{R}$. Such that:

$$X : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

$\forall\ ]a,b[$ in $\mathcal{B}(\mathbb{R})$: $X^{-1}(]a,b[) \in \mathcal{F}$.

In general a Random variable taking its values in a non ~~empty~~ empty set S is a mapping:

$$X : (\Omega, \mathcal{F}) \longrightarrow (S, \mathcal{B}(S)).$$

Such that $\forall B \in \mathcal{B}(S)$. $X^{-1}(B) \in \mathcal{F}$.

Examples: consider a Die (cube).

- X : The number appearing in the upper face:

$$S = X(\Omega) = \{1, 2, 3, 4, 5, 6\}:$$

$$\mathcal{B}(S) = \mathcal{P}(S)$$

- $X : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow (I, \mathcal{B}(I)) \quad (I = [-1, 1])$

$$\omega \longmapsto X(\omega) = \sin(\omega)$$

$B =]-\frac{1}{2}, 0[$; $X^{-1}((-\frac{1}{2}, 0)) = \{\omega \in \mathbb{R}$ s.t. $X(\omega) \in]-\frac{1}{2}, 0[\}$

$= \{\omega \in \mathbb{R}$ s.t. $\sin(\omega) \in]-\frac{1}{2}, 0[\}$

$]a,b[\subset [-1, 1]$. $X^{-1}(a,b) \overset{?}{\in} \mathcal{B}(\mathbb{R})$?

$\omega \to \sin(\omega)$ is a continuous function the ~~the set~~ set $\sin^{-1}(a,b)$ is a union of intervals then it belongs to $\mathcal{B}(\mathbb{R})$, Hence X is a random variable.

The mapping $X : (\Omega, \mathcal{F}) \longrightarrow (S, \mathcal{B}(S))$

$\omega \in \Omega \longmapsto X(\omega) \in S$.

is a random variable if for any set $B \in \mathcal{B}(S)$ $X^{-1}(B) \in \mathcal{F}$.

$X^{-1}(B) \subset \Omega$ is defined as follows.

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} = \underbrace{\{X \in B\}}_{\text{probability notation}}.$$

where $\mathcal{B}(S)$ is the Borel σ-algebra of S. that means.

~~$\mathcal{B}(S)$ is the set of all open sets of S~~.

$\cancel{\mathcal{B}(S)}$ $\mathcal{B}(S)$ is the σ-algebra generated by all open sets of S.

Examples: ① $S = \{a, b, c\}$. $\mathcal{B}(S) = \mathcal{P}(S)$.

② $S = [0, 4]$, $\mathcal{B}([0,4]) = \sigma(\text{I interval } \subset [0,4])$

③ $S = \mathbb{R}$ or $\mathbb{R}^d$, $\mathcal{B}(\mathbb{R}^d)$ is the σ-algebra generated by all open sets of $\mathbb{R}^d$.

④ $A \subset \Omega$, $\mathcal{F}_A = \{\emptyset, A, A^c, \Omega\}$ this is the σ-algebra generated by A is denoted $\sigma(A)$

$A, B \subset \Omega$, $\mathcal{F}^{A,B} = \sigma(A, B)$

$$\sigma(A,B) = \{\emptyset, \Omega, A, A^c, B, B^c, A\cap B, A\cap B^c, A^c\cap B, A^c \cap B^c, A\cup B, A^c\cup B^c, A^c\cup B, A\cup B^c\}$$

If $B_1, ---, B_m$ is a familly of sets of Ω

$$\mathcal{F}_m = \sigma(B_1, B_2, \dots, B_m).$$

W3-P13

A review of some homework questions: $\boxed{\text{Exercices}}$

- $Var(aX) = E[(aX)^2] - (E[aX])^2 = E[a^2X^2] - (aE[X])^2$
$= a^2(E[X^2] - (E[X])^2) = a^2 Var(X)$.

- $Var(X+b) = E[(b+X-E[b+X])^2] = E[(X-E[X])^2]$.

Let $X \hookrightarrow G(p)$: $\{\mathbb{N}^*;\ p(1-p)^{k-1},\ k \geq 1\}$. $0<p<1$

$$E[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1} \quad \text{set } q = 1-p.$$

$$E[X] = p\sum_{k=1}^{\infty} kq^{k-1} = p\sum_{k=1}^{\infty}(q^k)' = p\sum_{k=0}^{\infty}(q^k)' = p\left(\sum_{k=0}^{\infty} q^k\right)'$$

$$= p\left(\frac{1}{1-q}\right)' = p\frac{1}{(1-q)^2} = \frac{1}{p}.$$

Other method:

$$E[X] = \sum_{k=1}^{\infty} k(1-q)q^{k-1} = \sum_{k=1}^{\infty} kq^{k-1} - \sum_{k=1}^{\infty} kq^k$$

$$= \sum_{j=0}^{\infty}(j+1)q^j \quad \_\_\_\_ \quad \sum_{j=0}^{\infty} jq^j$$

$$= \sum_{j=0}^{\infty}(j+1)q^j - \sum_{j=0}^{\infty} jq^j.$$

$$E[X] = \sum_{k=1}^{\infty} k(1-q)q^{k-1} = \sum_{k=1}^{\infty}\left(kq^{k-1} - kqq^{k-1}\right).$$

$$= \sum_{k=1}^{\infty}\left(kq^{k-1}\right) - \sum_{k=1}^{\infty} kq^k \qquad \Bigg| \; j = k-1 \;\Downarrow\; k = j+1$$

$$= \sum_{j=0}^{\infty}(j+1)q^j - \sum_{j=0}^{\infty} jq^j \qquad \Big| \; j = k$$

$$= \sum_{j=0}^{\infty} q^j = \frac{1}{1-q} = \frac{1}{p} \; \square$$

Exercises:

$X \hookrightarrow \mathcal{G}(p)$. Find $E[X(X-1)]$ and deduce Var

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)pq^{k-1} = pq\sum_{k=2}^{\infty} k(k-1)q^{k-2}$$

$$= pq \sum_{k=2}^{\infty} (q^k)'' = pq \sum_{k=0}^{\infty} (q^k)''$$

$$= pq\left(\sum_{k=0}^{\infty} q^k\right)'' = pq\left(\frac{1}{1-q}\right)''$$

$$= pq\left[\left(\frac{1}{1-q}\right)'\right]' = pq\left[\frac{1}{(1-q)^2}\right]'$$

$2(1-p)+p$

$2-p$

$$E[X^2]-[EX] = pq\frac{2}{(1-q)^3} = \frac{2q}{p^2}$$

Then

$$E[X^2] = \frac{2q}{p^2} + \frac{p}{p^2} = \cancel{\frac{2}{p^2} - \frac{2}{p} + \frac{1}{p}} = \frac{2-p}{p^2}$$

$$Var(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}.$$

Problem (1)

	Young	Young + married	Young + single
Total	3000	1400	1600
Males	1320	600	1320 − 600 = 720
Females	1680	800	1680 − 800 = 880

The total number of young, femals single is 880.

Problem (2):

$P = P$ (Group watched none of the 3 sports).

$= 1 - P(WG \text{ or } WB \text{ or } WS) = 1 - 0.48 = 52\%$

let us first compute $P(WG \text{ or } WB \text{ or } WS) =$

$P(WG) + P(WB) + P(WS) - P(WG \cap WB) - P(WG \cap WS) - P(WB \cap WS)$
$+ P(WG \cap WB \cap WS) = 0.28 + 0.29 + 0.19 - 0.14 - 0.10 - 0.12$
$+ 0.0.8 = 0.48$

<u>Problem 3:</u> Denote by R : the event that a policyholder renews <u>at least</u> one policy.

We would like to compute $\mathbb{P}(R)$?!

$\mathbb{P}(R) = \mathbb{P}($ Renew only A, or renew only H, or renew $A \cap H)$

A : the event that a policyholder has an auto policy.

H : " " " " " " " " homeowner policy.

$$\mathbb{P}(R) = \mathbb{P}(\underbrace{R \cap A \cap H^c}_{A_1} \text{ or } \underbrace{R \cap A^c \cap H}_{A_2} \text{ or } \underbrace{R \cap A \cap H}_{A_3}).$$

A_1, A_2 and A_3 are pairewise disjoint.

~~Then~~ $A_1 \cap A_2 = R \cap A \cap H^c \cap A^c \cap H = \emptyset = A_2 \cap A_3 = A_1 \cap A_3$

$$\mathbb{P}(R) = \mathbb{P}(R \cap A \cap H^c) + \mathbb{P}(R \cap A^c \cap H) + \mathbb{P}(R \cap A \cap H).$$

$$= \mathbb{P}(R | A \cap H^c)\mathbb{P}(A \cap H^c) + \mathbb{P}(R | A^c \cap H)\mathbb{P}(A^c \cap H) + \mathbb{P}(R | A \cap H)\mathbb{P}(A \cap H).$$

$$\mathbb{P}(A \cap H^c) = \mathbb{P}(A) - \mathbb{P}(A \cap H) = 0.65 - 0.15 = 0.50.$$

$$\mathbb{P}(A^c \cap H) = \mathbb{P}(H) - \mathbb{P}(A \cap H) = 0.50 - 0.15 = 0.35.$$

$$\mathbb{P}(R) = 0.4 \times 0.50 + 0.6 \times 0.35 + 0.8 \times 0.15 = 53\%.$$

The end.

W3-P16

$(\Omega, \mathcal{F})$ $\boxed{\text{Lecture}}$

Ω: non-empty set

$\mathcal{F} \longrightarrow$ sigma-field or sigma-algebre. σ-field or σ-algebra: that is $\Omega \in \mathcal{F}$ and stable by ~~comple~~ complentation and ~~by~~ stable by contably infinite union.

A mapping from $(\Omega, \mathcal{F})$ to an other measurable space $(S, \mathcal{B}(S))$ is called a random variable. usually denoted by capital letters: $X, Y, Z, \ldots$

~~$\Omega = [-\pi, \pi]$, $\sin: [0,1] \longrightarrow$~~

$\Omega = [-\pi, \pi]$; $\sin: [-\pi, \pi] \longrightarrow [-1, 1]$.

$\omega \longmapsto \sin(\omega)$.

$B = [0, 1]$, $\sin^{-1}([0,1]) \ni \underline{2\pi}$

$\sin(2\pi) = 0$ but $2\pi \notin [-\pi, \pi]$ in general

this not a ~~ra~~ r.v.

but $\sin: ([-\pi, \pi], \mathcal{B}([-\pi, \pi])) \longrightarrow ([-1,1], \mathcal{B}([-1,1]))$

is ~~a~~ measurable mapping hence is a r.v.

<u>Some notations:</u> For a given subset B of $\mathbb{R}$.

$\{X \in B\} \stackrel{\text{def}}{=} \{\omega \in \Omega : X(\omega) \in B\}$ where $(\Omega, \mathcal{F})$ is a measurable space. Then $\{X \in B\} \in \mathcal{F}$ and X is a random variable: $(\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

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The distribution $\mathbb{P}_X$ of a r.v. X is the mapping:

$$\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \longrightarrow [0,1]$$
$$B \longmapsto \mathbb{P}_X(B) = \mathbb{P}(X \in B).$$

in particular: if $B = [a,b]$: $\mathbb{P}_X([a,b]) = \mathbb{P}(X \in [a,b])$
$= \mathbb{P}(a \leq X \leq b)$.

if $B =]-\infty, x]$ for any $x \in \mathbb{R}$:

$\mathbb{P}_X(]-\infty, x]) = \mathbb{P}(X \leq x) =: F_X(x)$ and F_X is called the cumulative distribution function (c.d.f.):

<u>Remark:</u> the c.d.f. F_X is defined in the whole space $\mathbb{R}$.

<u>Properties:</u> ① $x \longrightarrow F_X(x)$ is right continuous and limited from the left.

② $x \longrightarrow F_X(x)$ is not a decreasing function.

③ $\lim\limits_{x \to -\infty} F_X(x) = 0$ and $\lim\limits_{x \to +\infty} F_X(x) = 1$.

Set $S(x) = 1 - F_X(x) = \overline{F}_X(x)$. So $\overline{F}_X$ is called the survival function ~~of the~~ or the tail of F_X.

<u>Comment:</u> $\lim\limits_{x \to -\infty} F_X(x) = \mathbb{P}(X \leq -\infty)$, $\forall \omega \in \Omega$, $X(\omega) \in \mathbb{R}$.

the set $\{X \leq -\infty\} = \{\omega \in \Omega \mid X(\omega) \leq -\infty\} = \emptyset$.

$\lim\limits_{x \to +\infty} F_X(x) = \mathbb{P}(X < +\infty)$; $\{X < \infty\} = \{\omega \in \Omega : X(\omega) < +\infty\}$
$= \{\omega \in \Omega \; X(\omega) \in \mathbb{R}\}$
$= \Omega$.

We say that a distribution P_X is concentrated on the set B. if $P_X(B) = 1$ that $P(X \in B) = 1$.

In actuarial aplications, a non-negative r.v. is freqently called a risk.

There are two important particular types of r.v.: discrete and continuous.

We say that X is discrete iff there exists a denumerable subset $S = \{x_0, x_1, x_2, \ldots\}$ of $\mathbb{R}$ such that $P(X \in S) = 1$. In this case, we define the p.m.f. $p.f: S \longrightarrow [0,1]$ by: $f(x_k) = P(X = x_k)$; $k \geq 0$. The pair (S, f) gives a full probabilistic description of X.

$A, B, C \in \mathcal{F}$ such $P(C) > 0$.

$$P(A \cup B \mid C) = P(A|C) + P(B|C) - P(A \cap B|C).$$

$Q_c(A) = P(A|C)$; Q_c is also a probability:

$$Q_c(A \cup B) = Q_c(A) + Q_c(B) - Q_c(A \cap B).$$

$$P(A^c/B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - \frac{P(A \cap B)}{P(B)}$$

$$= 1 - P(A/B)$$

$P(A_i \cap A_j) = P(A_i)P(A_j)$ $\forall i \neq j$

$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$ $i \neq j \neq k$.

$(A_1, A_2, A_3, A_4, A_5)$.

$P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}) = \prod_{j=1}^{4} P(A_{i_j})$