① $R = R \cap \Omega$, because $\cancel{B}$ $R \subset \Omega$.

② $\Omega = \bigcup_{i=1}^{3} B_i = B_1 \cup B_2 \cup B_3$. B_i are disjoint events.

Then $R = R \cap \left(\bigcup_{i=1}^{3} B_i\right) = \bigcup_{i=1}^{3} (R \cap B_i)$.

$$P(R) = P(\underbrace{R\cap B_1} \cup \underbrace{R \cap B_2} \cup \underbrace{R\cap B_3})$$
$$= P(R\cap B_1) + P(R\cap B_2) + P(R \cap B_3).$$
$$= P(B_1)P(R|B_1) + P(B_2)P(R|R_2) + P(B_3)P(R/B_3).$$

B_1: $1R + 4W$

B_2: $1R + 2W$

B_3: $5R + 4W$

$P(B_1) = \frac{1}{3}$, $P(B_2) = \frac{1}{6}$ and $P(B_3) = \frac{1}{2}$

Remark first that $P(B_1) + P(B_2) + P(B_3) = 1$.

<u>Find</u> $P(B_1|R) = \dfrac{P(B_1 \cap R)}{P(R)} = \dfrac{P(B_1)P(R|B_1)}{P(R)}$.

details are in the pdf on the Bb (lms).

Exercises :

the number of the extracted bull.

P4 ① X : the number of the extracted bull. X has a uniform distribution in S : it is given $(k, \frac{1}{N})$ $k=1,2,\dots,N$

$S_X = S = \{1, \dots, N\}$.

$E[X] = \frac{N+1}{2}$ and $Var(X) = \frac{N^2-1}{12}$.

② $S_X = \{0,1,\dots\} = \mathbb{N}$. , $a \in \mathbb{R}_+^*$.

ⓐ $\forall k \in \mathbb{N}^*$ we have $P(X=k) = \frac{a}{k} P(X=k-1)$.

We need to find $P(X=k)$, $k \in \mathbb{N}$?!

$$\boxed{P(X=k) = \frac{a}{k} P(X=k-1)} = \frac{a}{k} \cdot \frac{a}{k-1} P(X=k-2) \dots$$

$$= \frac{a^k}{k!} P(X=0). \quad k \geq 1.$$

$$\boxed{\sum_{k=1}^{\infty} P(X=k) < 1 \qquad P(X \in \mathbb{N}^*) < 1.}$$

$$P(X \in \mathbb{N}) = 1$$

$k=1$: $\times P(\cancel{X=1}) = \frac{a}{1} P(X=0)$ $\times$

$k=2$ $\times P(\cancel{X=2}) = \frac{a}{2} P(\cancel{X=1})$ $\times$

$\times P(\cancel{X=3}) = \frac{a}{3} P(\cancel{X=2})$ $\times$

$\times$

$\times P(\cancel{X=n-1}) = \frac{a}{n-1} P(\cancel{X=n-2})$ $\times$

$\times P(X=n) = \frac{a}{n} P(\cancel{X=n-1})$.

$$P(X=n) = \frac{a^n}{n!} P(X=0). \quad n \geq 1.$$

$P(X \in \mathbb{N}) = 1$

$$\sum_{n=0}^{\infty} P(X=n) = 1 = P(X=0) + P(X=0) \sum_{n=1}^{\infty} \frac{a^n}{n!}.$$

$$\Leftrightarrow 1 = P(X=0)\left(1 + \sum_{n=1}^{\infty} \frac{a^n}{n!}\right)$$

$$= P(X=0)\left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right) = P(X=0) e^a.$$

Then: $P(X=0) = e^{-a}$

and $n \geq 1$ $P(X=n) = e^{-a}\frac{a^n}{n!}$ $\Bigg\}$ $\forall n \in \mathbb{N}\ P(X=n) = \frac{a^n}{n!}e^{-a}$,

b): The distribution of X is Poisson with parameter a.

3). ~~From the~~ Let X be the r.v. describing the ~~number~~ number of claims during the five-year period.

$P_n = P(X=n)$. We have $\boxed{P_{n+1} = \frac{1}{4}P_n}$.

$$\left.\begin{aligned} \cancel{P_1} &= P_0\frac{1}{4} \\ \cancel{P_2} &= \cancel{P_1}\frac{1}{4} \\ &\vdots \\ P_n &= \cancel{P_{n-1}}\frac{1}{4} \end{aligned}\right\} \Longrightarrow \boxed{P_n = P_0\frac{1}{4^n}}$$

The desired probability is $P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - P(X=0) - P(X=1)$.

$= 1 - P_0 - P_1 = 1 - P_0 - \frac{1}{4}P_0 = 1 - P_0\frac{5}{4}$.

We know $P(X \in \mathbb{N}) = 1 = P_0 + P_1 + P_2 + \cdots$

That is $\boxed{\sum_{n=0}^{\infty} P_n = 1} \Leftrightarrow \sum_{n=0}^{\infty} P_0\frac{1}{4^n} = 1$

$\Leftrightarrow P_0 \cdot \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = 1 \Leftrightarrow P_0\frac{1}{1-\frac{1}{4}} = 1$

$P_0 = \frac{3}{4}$. So $P(X \geq 2) = 1 - \frac{15}{16} = \frac{1}{16}$.

W4. P22

P5:

(1) The expected payment of hospitalization is given:

$$100\big(P(X=1)+P(X=2)+P(X=3)\big)+50\big(P(X=4)+P(X=5)\big)$$

$$=100\left(\frac{1\times5}{3\times5}+\frac{4}{15}+\frac{1\times3}{5\times3}\right)+50\left(\frac{2}{15}+\frac{1}{15}\right)$$

$$=\frac{1200}{15}+\frac{150}{15}=\frac{1350}{15}=90.$$

(2) Let X be a r.v: $S_X=\mathbb{N}$: $P(X=n)=\frac{1}{(n+1)(n+2)}$.

verify the $\sum_{n=0}^{\infty}P(X=n)=1$...

$$P(X=n)=\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$

$$\sum_{n\geq0}\frac{1}{(n+1)(n+2)}=\sum_{n\geq0}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=1$$

$$=1-\cancel{\tfrac{1}{2}}+\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}+\cdots+\frac{1}{n}-\frac{1}{n+1}+\cdots$$

$$=1.$$

a) $$P(X\geq1\mid X\leq4)=\frac{P(1\leq X\leq4)}{P(X\leq4)}=\frac{\sum_{k=1}^{4}P(X=k)}{\sum_{k=0}^{4}P(X=k)}$$

$$\cancel{\sum_{k=1}^{j}P(X=k)=\frac{1}{1+1}-\frac{1}{j+2}}=\frac{\sum_{k=1}^{4}P(X=k)}{P(X=0)+\sum_{k=1}^{4}P(X=k)}$$

$$\sum_{k=1}^{n}P(X=k)=\frac{1}{2}-\frac{1}{n+2}\quad\left(n=4:\ \frac{3}{2\times3}-\frac{1}{6}=\frac{1}{3}\right.$$

$$=\frac{1}{2}-\cancel{\tfrac{1}{3}}+\cancel{\tfrac{1}{3}}-\frac{1}{4}+\cdots\cancel{\tfrac{1}{n-1}}-\cancel{\tfrac{1}{n}}+\cancel{\tfrac{1}{n}}-\cancel{\tfrac{1}{n+1}}+\cancel{\tfrac{1}{n+1}}-\frac{1}{n+2}$$

$$P(X\geq1\mid X\leq4)=\frac{1/3}{\frac{1}{2}+1/3}=\frac{6}{15}=\frac{2}{5}.$$

Some examples: If X is Poisson r.v. (λ) that is:

- $S_X = \mathbb{N}$, $\forall k \in \mathbb{N}$ $P(X=k) = f(k) = e^{-\lambda} \frac{\lambda^k}{k!}$.

- $E[X] = \sum_{x=0}^{\infty} f(x)x = \sum_{n=0}^{\infty} n f(n) = e^{-\lambda} \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!}$.

$$= e^{-\lambda} \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$$\boxed{n! = n(n-1)(n-2)!}$$

- $E[X(X-1)] = E[X^2] - E[X] = E[X^2] - \lambda$.

Def: $\parallel$

$$\sum_{n=0}^{\infty} n(n-1)f(n) = \sum_{n=0}^{\infty} n(n-1) e^{-\lambda} \frac{\lambda^n}{n!} = \sum_{n=2}^{\infty} e^{-\lambda} \frac{\lambda^{n-2}\lambda^2}{(n-2)!}$$

$$= \lambda^2 e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!} = \lambda^2 e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda^2 e^{-\lambda} e^{\lambda}.$$

$$= \lambda^2.$$

Deduce $E[X^2] = \lambda^2 + \lambda$.

Then $Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

standard deviation of X.

$\sigma = \sigma_X = \sqrt{Var(X)}$.

If $X \hookrightarrow \mathcal{P}(\lambda)$. $\begin{cases} E[X] = \lambda \\ \sigma = \sigma_X = sd(X) = \sqrt{\lambda} \end{cases}$ $\boxed{\lambda > 0}$

MGF or mgf: $M_X(t) = M(t) = E[e^{tX}]$

$M'(t) = E[Xe^{tX}] \Rightarrow M'(0) = E[X]$.

$M''(t) = E[X^2 e^{tX}] \Rightarrow M''(0) = E[X^2]$.

$Var(X) = M''(0) - (M'(0))^2$.

If X is continuous real r.v. then $S_X \subseteq \mathbb{R}$.

Examples: $X \hookrightarrow \exp(\lambda)$: • $S_X =]0, +\infty[$. $\lambda > 0$

•• $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

If f is a pdf of a r.v. then:

① $\int_S f(x)\,dx = 1$ and ② $\forall x \in S$. $f(x) \geq 0$.

If $X \hookrightarrow \exp(\lambda)$, $\lambda > 0$.

$$E[X] = \int_0^{+\infty} x f(x)\,dx = \int_0^{+\infty} x \lambda e^{-\lambda x}\,dx = \cancel{[-xe^{-\lambda x}]_0^{+\infty}}$$

$$= [-xe^{-\lambda x}]_0^{\infty} + \int_0^{+\infty} e^{-\lambda x}\,dx \qquad \boxed{\lambda e^{-\lambda x} = [-e^{-\lambda x}]'}$$

$$= 0 + \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^{+\infty} = \frac{1}{\lambda}.$$

$$E[X^2] = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\,dx = \int_0^{+\infty} x^2 [-e^{-\lambda x}]'\,dx$$

$\int u(x) v'(x)\,dx = [uv] - \int u'v\,dx$

$$= [-x^2 e^{-\lambda x}]_0^{+\infty} + \int_0^{+\infty} 2x e^{-\lambda x}\,dx$$

$$= \frac{2}{\lambda} \int_0^{+\infty} x \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda} E[X] = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \Rightarrow sd(X) = \sigma_X = \frac{1}{\lambda}.$$

Let $\mu = E[X]$, $\sigma^2 = Var(X)$.

. Find $E\left[\frac{X-\mu}{\sigma}\right]$ and $Var\left(\frac{X-\mu}{\sigma}\right)$.

$$E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma} E[X-\mu] = \frac{1}{\sigma}\left[E[X] - \mu\right] = 0$$

$$Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X-\mu) = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1$$

Exercises: Find the MGF or mgf of the following r.v: $X \hookrightarrow B(n,p)$, $X \hookrightarrow G(p)$, $X \hookrightarrow \mathcal{P}(\lambda)$

$X \hookrightarrow \mathcal{U}([a,b])$; $X \hookrightarrow exp(\lambda)$.

$X \hookrightarrow \mathcal{N}(\mu, \sigma^2)$.

Deduce the moments of X in each case. expection - variance, skeness, kurtosis.

Chap2: Multivariate random variables:

Motivation: $Z = X + Y$ $\rightarrow$ describe a loss of an insurance compagnie. Some times ~~the~~ ~~one~~ I. C. is interested to estimate the expected loss and also to calculate the probablity that this loss is less than 1 Million: Then we need to evaluate:

- $E[Z]$ and $P(Z \leq 1)$.

In fact $E[Z] = E[X] + E[Y]$ | we can write this for any X and Y:

If we know the distribution Z. $E[Z] = \begin{cases} \int_{S_Z} z f(z) dz \\ \sum_{z \in S_Z} z f(z) \end{cases}$

to Do this we need the distribution of (X, Y).

and $P(Z \leq 1) = P(X+Y \leq 1)$. So we need

$\{X+Y \leq 1\} \overset{\neq}{\subset} \{X \leq 1, Y \leq 1\}$

the distribution (X, Y) (joint distribution).

Def: let X and Y be 2 r.v. let S denote: the corespond two-dimensional space of X and Y. set $f(x,y) = P(X = x, Y = y)$ for $x, y \in S = S_X \times S_Y$.

then: $S_{(X,Y)} = S_X \times S_Y = S$. X and Y are two discrete r.v.

~~the~~

W4.P27

The function $f(x,y)$ is called joint probability mass function of (X,Y) and has the following properties:

a. $0 \leq f(x,y) \leq 1$.

b. $\sum_{S} f(x,y) = 1$.

c. $\forall A \subset S$: $P((X,Y) \in A) = \sum_{x,y \in A} f(x,y)$.

<u>Example</u>: Roll a pair of fair dice. $\# S = 36$

$S = \{ (i,j) \; i,j \in \{1,2,\dots,6\} \}$.

Let X_1 be the r.v corresponding to the number appearing on the upper face of dice 1.

and X_2 similarly.

Now consider: $X = \min(X_1, X_2)$ and $Y = \max(X_1, X_2)$.

Find the distributions of X_1, X_2, (X_1, X_2), (X,Y).

(1) X_1 is uniform on $\{1,2,3,4,5,6\}$ $dist(X_1) = (k, \frac{1}{6})$ $k = \{1,\dots,6\}$.

X_2 " " " " " " " " " " " " " " " " "

(2) $Dist(X_1 X_2) = \{ ((i,j); \frac{1}{36}) \; i,j \in \{1,2,\dots,6\} \}$ is also uniform dis.

(3) $S_{(X,Y)} = \{ (i,j) \; ; \; i,j \in \{1,2,\dots,6\} \}$.

<u>example</u> $(X,Y) = (3,2) = \{ X = 3, Y = 2 \}$.

~~X_1, X_2 $X_1 = 3$ X_2~~ $(X_1, X_2) = (3,2) \Leftrightarrow \{ X_1 = 3, X_2 = 2)$.

<u>then</u>: $X = 2$ and $Y = 3$.

W4 P28

$(X,Y)=(2,3) \Leftrightarrow (X_1=2,\ X_2=3)$ or $(X_1=3,\ X_2=2)$.

$$\begin{aligned} P((X,Y)=(2,3)) &= P((X_1,X_2)=(2,3)) + P((X_1,X_2)=(3,2)). \\ &= P(X_1=2, X_2=3) + P(X_1=3, X_2=2) \\ \text{(independence)}\quad &= P(X_1=2)P(X_2=3) + P(X_1=3)P(X_2=2) \\ &= \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6}. \end{aligned}$$

$P((X,Y)=(2,2))$

$$P(X=2, Y=2) = P(X_1=2, X_2=2) = P(X_1=2)P(X_2=2) = \frac{1}{6}\cdot\frac{1}{6}.$$

In general the joint pmf of (X,Y) is given by.

$$f(i,j) = \begin{cases} \frac{1}{36} & \text{if } 1\le i=j\le 6. \\ \frac{2}{36} & \text{if } 1\le i<j\le 6. \end{cases}$$

W4.29