Solving some exercises on the homework (HW2)

P3. $X \hookrightarrow \mathcal{P}(\lambda)$. $Y = \begin{cases} \frac{X}{2} & \text{if } X \text{ is even} \\ 0 & \text{if } X \text{ is odd} \end{cases}$

$Y$ follows a mixed distribution.

Find the distribution of $Y$?

Find the distribution, $f(0)$ and $f(n)$; $n \geq 1$.

• $S_Y = \mathbb{N}$; p.m.f. $f(n)$, $n \geq 0$,

We have

$f(0) = P(Y=0) = P(X \text{ is odd}) \cancel{= P(X \leq 2k+1)}$

$= P(X=1 \text{ or } X=3 \text{ or } X=5, \ldots, \text{ or } X=2n+1 \text{ or} \ldots)$

$= P\left(\bigcup_{n\geq 0} \underbrace{\{X=2n+1\}}_{A_n}\right)$.

We know

$$P\left(\bigcup_{n\geq 0} A_n\right) = \sum_{n\geq 0} P(A_n) \text{ ; whenever } A_i \cap A_j = \phi \ \forall i \neq j.$$

Since $A_n \cap A_m = \phi \ \forall n \neq m$.

$\{X=2n+1\} \cap \{X=2m+1\} = \phi$.

$$f(0) = \sum_{n\geq 0} P(X=2n+1) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^{2n+1}}{(2n+1)!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} = e^{-\lambda} sh(\lambda) = e^{-\lambda}\left(\frac{e^{\lambda}-e^{-\lambda}}{2}\right).$$

$$= \frac{1-e^{-2\lambda}}{2}.$$

Now $f(n)$ $n \geq 1$, $f(n) = P(Y=n) = P(Y=n, X \text{ is even})$

$= P\left(\frac{X}{2}=n\right)$.

$= P(X=2n)$.

So $n \geq 1$ $f(n) = \mathbb{P}(X = 2n) = e^{-\lambda} \dfrac{\lambda^{2n}}{(2n)!}$.

c/c: The distribution of $Y$ is given by: $(S_Y, f(n))_{n \in S_Y}$:

$S_Y = \mathbb{N}$ and $\begin{cases} f(0) = \dfrac{1 - e^{-2\lambda}}{2} \\ f(n) = e^{-\lambda} \dfrac{\lambda^{2n}}{(2n)!} \text{ for all } n \geq 1. \end{cases}$

$$\frac{k}{2k(2k-1)(2k-2)\cdots 2}$$

$E[Y] \neq E[\frac{X}{2}]$ is not true because $Y \neq \frac{X}{2}$.

$$E[Y] = \sum_{k=0}^{\infty} k f(k) = \sum_{k=1}^{\infty} k f(k) = \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^{2k}}{(2k)!}$$

$$= e^{-\lambda} \frac{\lambda}{2} \underline{\sum_{k=1}^{\infty} \frac{\lambda^{2k-1}}{(2k-1)!}} = \frac{\lambda}{2} e^{-\lambda} \left( \frac{e^{\lambda} - e^{-\lambda}}{2} \right).$$

$$\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = \frac{e^{\lambda} + e^{-\lambda}}{2} \text{ and } \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} = \frac{e^{\lambda} - e^{-\lambda}}{2}.$$

We can do the same calculation to $\boxed{E[Y^2]}$

$$E[2Y(2Y-1)] = \sum_{k=0}^{\infty} 2k(2k-1) \frac{\lambda^{2k}}{(2k)!} e^{-\lambda} \text{ and so on}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^2 \lambda^{2k-2}}{(2k-2)!} e^{-\lambda} = \lambda^2 e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{2k-2}}{(2k-2)!}$$

$\boxed{2k - 2 = 2j}$

$$4E[Y^2] - 2E[Y] = \lambda^2 e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!}$$

$$= \lambda^2 e^{-\lambda} \frac{e^{\lambda} + e^{-\lambda}}{2}.$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$$

$\mathbb{N}^* = \{n, n \geq 1\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$

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Homework solutions continued.

P3. Q2. Let $X$ and $Y$ be two $\perp\!\!\!\perp$ r.v taking their values in $\mathbb{N}$.

(*) Recall the definition of $\perp\!\!\!\perp$ of $X$ and $Y$.

$$X \perp\!\!\!\perp Y \Leftrightarrow F_{(X,Y)}(x,y) = F_X(x) F_Y(y).$$

$$\Leftrightarrow \mathbb{P}(X=x, Y=y) = P(X=x)P(Y=y) \text{ for all } x,y \in \mathbb{N}.$$

$p = \mathbb{P}(X=1)$

$X \hookrightarrow Br(p)$, $Y \hookrightarrow \mathcal{P}(\lambda)$. $Z = XY$.

a. Distribution Z:

$Z$ is discrete r.v. because it takes its values in $\mathbb{N}$

- $S_Z = \mathbb{N}$, p.m.f: $f(z)$, $z \in \mathbb{N}$. | $f(z) = \mathbb{P}(Z=z)$.

We have $f(z) = \mathbb{P}(XY = z) = \mathbb{P}(XY = z, \Omega)$.

$\Omega = \{X=0\} \cup \{X=1\}$, then:

$$f(z) = \mathbb{P}(\{XY = z\} \cap \{X=0\} \cup \{X=1\})$$

~~Explanation~~ $= \mathbb{P}(XY = z, X=0) + \mathbb{P}(XY = z, X=1)$

$= \mathbb{P}(0 = z, X=0) + \mathbb{P}(Y = z, X=1)$.

- $\boxed{z=0}$

$$f(0) = \mathbb{P}(XY=0) = \mathbb{P}(X=0 \text{ or } Y=0)$$

$= P(X=0) + P(Y=0) - P(X=0, Y=0)$.

$= P(X=0) + P(Y=0) - \mathbb{P}(X=0)P(Y=0) \leftrightarrow X \perp\!\!\!\perp Y$.

$= 1-p + e^{-\lambda} - (1-p)e^{-\lambda} = 1-p+pe^{-\lambda}$.

Sol HK2 continued.

$$\forall z \geq 1 : f(z) = \underbrace{\overset{\Phi}{\overbrace{P(0=z, X=0)}}}_{\underset{0}{\parallel}} + P(Y=z, X=1)$$

$$= P(Y=z)P(X=1) \quad X \perp\!\!\!\perp Y$$

$$= p e^{-\lambda} \frac{\lambda^z}{z!}$$

c/c: p.m.f of $Z$: $f(z) = \begin{cases} 1-p+pe^{-\lambda} & z=0 \\ pe^{-\lambda}\frac{\lambda^z}{z!} & z \geq 1 \end{cases}$

- $E[Z] = E[XY] = E[X]E[Y]$ because $X \perp\!\!\!\perp Y$
$= p\lambda$

$$\underset{\parallel}{\phantom{x}} \sum_{z=0}^{\infty} z f(z) = \sum_{z=1}^{\infty} z f(z) = \sum_{z=1}^{\infty} z p e^{-\lambda} \frac{\lambda^z}{z!} = pe^{-\lambda}\sum_{z=1}^{\infty} \frac{z\lambda^z}{z!}$$

$$= p\lambda.$$

$$Var(Z) = E[Z^2] - (E[Z])^2 = E[X^2Y^2] - (E[X])^2(E[Y])^2$$

$$= E[X^2]\boxed{E[Y^2]} - p^2(E[Y])^2$$

$X \hookrightarrow Br(p)$: $E[X^{1996}] = 0^{1996}(1-p) + 1^{1996} \cdot p = p.$

$$\hookrightarrow Var(Z) = p\boxed{(Var(Y) + (E[Y])^2)} - p^2\lambda^2$$

$$= p(\lambda + \lambda^2) - p^2\lambda^2 = \boxed{p\lambda + p(1-p)\lambda^2.}$$

b). mgf of $Z$: $M_Z(t) = E[e^{tZ}]$.

WJ.P33

$$M_Z(t) = E[e^{tZ}] = \sum_{n=0}^{\infty} e^{tn} P(Z=n).$$

$$= P(Z=0) + \sum_{n=1}^{\infty} e^{tn} P(Z=n)$$

$$= 1-p+pe^{-\lambda} + \sum_{n=1}^{\infty} e^{tn}\, pe^{-\lambda}\frac{\lambda^n}{n!}$$

| b | p |
|---|---|

$$= 1-p+pe^{-\lambda} + pe^{-\lambda}\sum_{n=1}^{\infty} \frac{(\lambda e^t)^n}{n!}$$

$$= 1-p+pe^{-\lambda}\left(1+\sum_{n=1}^{\infty} \frac{(\lambda e^t)^n}{n!}\right)$$

$$= 1-p+pe^{-\lambda}\sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = 1-p+pe^{-\lambda}e^{\lambda e^t}$$

$$= 1-p+pe^{\lambda(e^t-1)}. \square$$

$M'_Z(t)$ and $M''_Z(t)$:

$$M'_Z(t) = pe^{-\lambda}e^{\lambda e^t}(\lambda e^t) \longrightarrow M'_Z(0) = pe^{-\lambda}e^{\lambda}\lambda = p\lambda.$$

$$M''_Z(t) = pe^{-\lambda}e^{\lambda e^t}(\lambda e^t)^2 + pe^{-\lambda}e^{\lambda e^t}\lambda e^t$$

$$M''_Z(0) = p\lambda^2 + p\lambda.$$

$$\boxed{P(X=1 \mid Z=3) = \frac{P(X=1, XY=3)}{P(Z=3)}}$$

using mgf $E[Z] = M'_Z(0) = p\lambda.$

$$\boxed{Var(Z)} = M''_Z(0) - (M'_Z(0))^2 = p\lambda^2 + p\lambda - p^2\lambda^2$$

$$\boxed{= p\lambda + p(1-p)\lambda^2.} \square$$

c. $$P(X=1 \mid Z=0) = \frac{P(X=1, Z=0)}{P(Z=0)} = \frac{P(X=1, XY=0)}{P(Z=0)}$$

$$= \frac{P(X=1, Y=0)}{P(Z=0)} \overset{\perp}{=} \frac{P(X=1)P(Y=0)}{P(Z=0)}$$

WS-P34

c.d.f : $F_X ; F : \mathbb{R} \longrightarrow [0,1]$.

$x \longmapsto F(x) = P(X \leq x)$.

$\boxed{P.S}$ Q 1

set $f(x) = \begin{cases} a(e^{-2x} - e^{-3x}) & x > 0 \\ 0 & \text{otherwise} \end{cases}$ with $a > 0$.

Question ① is $f$ a p.d.f ?

- $f(x) \geq 0 \ \forall x \in \mathbb{R}$ •• $\int_{-\infty}^{+\infty} f(x)\,dx = 1 \Leftrightarrow a = 6$.

yes

Question ② Find c.d.f. of $X$ where $X$ is r.v.

with p.d.f $f(x) = \begin{cases} 6(e^{-2x} - e^{-3x}) & \text{if } x > 0 \\ 0 & \text{elswhere} \end{cases}$

$F(x) = \int_{-\infty}^{x} f(y)\,dy$ — if $x \leq 0$ / if $x > 0$

$]-\infty, c] \subset ]-\infty, d]$. $\int_{-\infty}^{c} f(y)\,dy \leq \int_{-\infty}^{d} f(y)\,dy$

case 1: if $x \leq 0$, $F(x) = \int_{-\infty}^{x} f(y)\,dy \leq \int_{-\infty}^{0} f(y)\,dy = 0$.

case 2: if $x > 0$, $F(x) = \int_{0}^{x} f(y)\,dy = 1 + 2e^{-3x} - 3e^{-2x}$

c.d.f : $F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 + 2e^{-3x} - 3e^{-2x} & x > 0 \end{cases}$

Question ③ Calculate $P(X < 1)$?

we have $P(X < 1) = F_X(1) = 1 + 2e^{-3} - 3e^{-2}$

$\lim_{x \to +\infty} F_X(x) = 1$.

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PS. Q2:

$$f(x) = \begin{cases} 25x & 0 < x < \frac{2}{10} \\ 1.5225(1-x) & \frac{2}{10} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

c.d.f: 4 cases:

(1) $x \leq 0$; $F_X(x) = 0$

(2) $0 < x < \frac{2}{10}$: $F_X(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{0} f(y)\,dy + \int_{0}^{x} f(y)\,dy$

$$F_X(x) = \int_0^x f(y)\,dy = \int_0^x 25y\,dy = \frac{25}{2}x^2.$$

(3) $\frac{2}{10} < x < 1$, $F_X(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{\frac{2}{10}} f(y)\,dy + \int_{\frac{2}{10}}^{x} f(y)\,dy$

$$= F_X\left(\frac{2}{10}\right) + \int_{2/10}^{x} 1.5225(1-y)\,dy \cancel{= \frac{1}{2} + \frac{1.5225}{2}\left(1-(1-x)^2\right)}$$

$$= \frac{1}{2} - \frac{1.5225}{2}\left[(1-y)^2\right]_{\frac{2}{10}}^{x}$$

$$\frac{10-2}{10} = \frac{4}{5}$$

$$= \frac{1}{2} - \frac{1.5225}{2}\left[(1-x)^2 - \frac{4^2}{5^2}\right]; \frac{2}{10} < x < 1$$

$$= \cancel{\frac{1}{2} + \frac{16}{25}} - \frac{1.5225}{2}\cancel{(1-x)^2}$$

(4) $x > 1$, $F_X(x) = 1$

$$\boxed{F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{25}{2}x^2 & 0 < x < \frac{2}{10} \\ -\frac{609}{800}x^2 + \frac{609}{400}x + \frac{4519}{20000} & \text{if } \frac{2}{10} < x < 1 \\ 1 & x \geq 1 \end{cases}}$$

WS-P36

P4.84] X single loss : $X \hookrightarrow \exp\left(\frac{2}{10^3}\right)$.

the density of X is given

$$f_X(x) = \begin{cases} \frac{2}{10^3} e^{-\frac{2}{10^3}x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

a) $E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^{+\infty} x \lambda e^{-\lambda x}\,dx \quad \left(\lambda = \frac{2}{10^3}\right).$

$$= \int_0^{+\infty} x\,(-e^{-\lambda x})'\,dx = \left[-x e^{-\lambda x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}\,dx.$$

$$= \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^{+\infty} = \frac{1}{\lambda} = \frac{10^3}{2}.$$

b) $E[X^2] = \int x^2 f(x)\,dx = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\,dx = \int_0^{+\infty} x^2 (-e^{-\lambda x})'\,dx.$

$$= \left[-x^2 e^{-\lambda x}\right]_0^{+\infty} + \int_0^{+\infty} 2x e^{-\lambda x}\,dx.$$

$$= \frac{2}{\lambda} \int_0^{+\infty} x \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda} E[X] = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$sd(X) = \sigma_X = \sqrt{\mathrm{Var}(X)} = \frac{1}{\lambda} = \frac{10^3}{2} = 500.$$

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an other example

Y is r.v. with pdf $f(x) = \begin{cases} \frac{1}{a} e^{-\frac{x}{a}} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$ $a > 0$

$E[Y] = a$.

WS-P37

8.5.1 A policy with a deductible $K$ for some loss:

$$Z = \underbrace{X-K}_{\text{insurance company}} + \underbrace{K}_{\text{policyholder}}$$

~~police~~

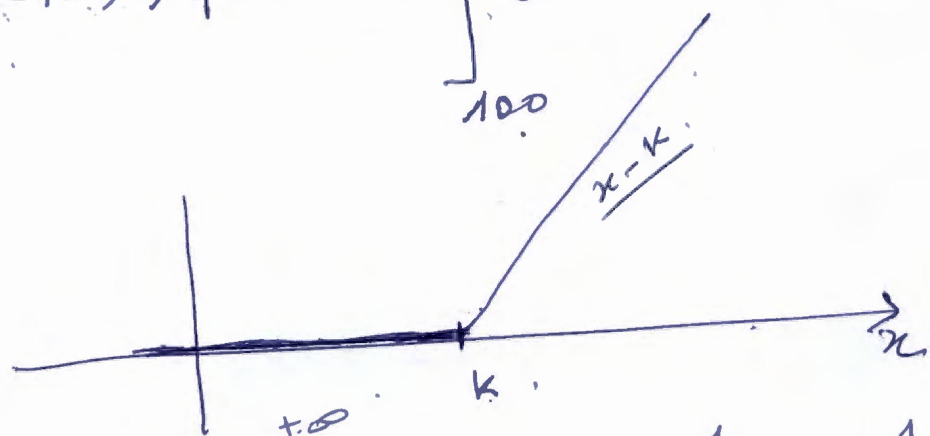
For the insurance company the claim is given by claim $:= \max(X-K, 0)$.
The expectation of the single claim is

$E[\max(X-K,0)]$ . $K = 100$.

$$E[\max(X-\cancel{1}00,0)] = E[g(X)] \qquad g(x) = (x-\cancel{100})^+$$

$$= \int_{-\infty}^{+\infty} \max(x-100,0)\, f(x)\, dx = \int_{100}^{+\infty} (x-100)\,\lambda e^{-\lambda x}\, dx$$

$\max(x-K,0)$




$$= \int_{100}^{+\infty} \lambda x e^{-\lambda x}\, dx - 100 \int_{100}^{+\infty} \lambda e^{-\lambda x}\, dx = 100 e^{-\frac{1}{5}} + 500 e^{-\frac{1}{5}} - 100 e^{-\frac{1}{5}}$$

$$= 500 e^{-\frac{1}{5}} . \boxed{\cdot}$$

$$\max(x-K,0) = \begin{cases} x-K & \text{if } x > K \\ 0 & \text{if } x \le K \end{cases}$$

Let $X$ and $Y$ be two r.v. of discrete type.

$S_X$ : the set of all possible values of $X$.

$S_Y$ : ——————— of $Y$.

$Z = (X, Y)$ is also a r.v. taking its values $\underset{\sim}{S}_Z = \underset{\sim}{S}_X \times S_Y = \underset{\sim}{S}$.

The p.m.f. of $Z$ is given: $f_Z(z)$, $z \in \underset{\sim}{S}_Z$

$\cancel{f_Z(z) = f_Z(x,y)} \neq f_Z(z) = P(Z = z)$

If $z = (x, y)$ ; $f_Z(z) = f_Z(x,y) = P((X,Y) = (x,y))$.

$= P(X = x, Y = y) = f_{(X,Y)}(x,y) = f(x,y)$.

c/ The distribution of $Z$ is completely determined by: • $\underset{\sim}{S}_Z$ and • $f_Z(z)$, $\forall z \in \underset{\sim}{S}_Z$.

<u>Marginal distributions:</u> $f_X(x)$ and $f_Y(y)$, $x \in S_X$, $y \in S_Y$

where $f_X(x) = \sum_{y \in S_Y} f_{(X,Y)}(x,y)$ and $f_Y(y) = \sum_{x \in S_X} f_{(X,Y)}(x,y)$.

<u>Conditional distributions:</u>

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \quad \text{and} \quad f_{Y|X}(y|x) = \frac{f_{(X,Y)}(x,y)}{f_X(x)}.$$

for continuous r.v. we juste replace the sum by the integral.

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## Conditional c.d.f:

$$F_{X|Y}(x|y) = P(X \le x \mid Y \le y) = \frac{P(X \le x, Y \le y)}{P(Y \le y)}.$$

$$= \frac{F_{(X,Y)}(x,y)}{F_Y(y)}.$$

**Examples:** $X$ and $Y$ r.v. such that $S_X = \{1,2,3\}$
$S_Y = \{1,2\}$. $Z = (X,Y)$, $S_Z = \{(i,j)\ i \in S_X, j \in S_Y\}$.

p.m.f of $(X,Y)$: $f_{(X,Y)}(x,y) = \frac{x+y}{21}$.

**Questions**: $f_X$, $f_Y$, $f_{X|Y}$ and $f_{Y|X}$:

$f_X(x) = \frac{2x+3}{21}$, $x = 1,2,3$. (marginal of $X$).

$f_Y(y) = \frac{6+3y}{21}$, $y = 1,2$ (marginal of $Y$).

$f_{X|Y}(x/y) = \frac{x+y}{3y+6}$; $x = 1,2,3$ and $y = 1,2$

$f_{Y|X}(y|x) = \frac{x+y}{2x+3}$, $x = 1,2,3$ and $y$ 1 2.

If: $f_{(X,Y)}(x,y) = f_X(x) f_Y(y) \Leftrightarrow X \perp\!\!\!\perp Y$.
$\forall\, x,y \in S_X \times S_Y$.

If $\exists\, x_0, y_0 \in S_X \times S_Y$ s.t. $f_{(X,Y)}(x_0,y_0) \ne f_X(x_0) f_Y(y_0)$ then $X$ and $Y$ are dependent.

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