Some exercises from the lecture notes.

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x < y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(*). verify that f is a joint p.d.f.

- we $f_{X,Y}(x,y) \geq 0 \quad \forall\, x,y \in \mathbb{R}\times\mathbb{R} = \mathbb{R}^2$.
- .. $\iint f(x,y)\,dx\,dy = 1$?? to verify.

$$\int_0^1 dy \int_0^y 2\,dx = \int_0^1 2y\,dy = y^2\Big]_0^1 = 1.$$

a) $$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = \iint_{\{x \le \frac{1}{2},\, y \le \frac{1}{2}\}} f(x,y)\,dx\,dy = \int_0^{1/2} dy \int_0^y 2\,dx = \frac{1}{4}$$

$$\begin{cases} 0 \le x < y \le 1 \\ x \le \frac{1}{2},\ y \le \frac{1}{2} \end{cases} = \left\{ 0 \le x < y \le \tfrac{1}{2} \right.$$

$$= \int_0^{1/2} dx \int_x^{1/2} 2\,dy = \int_0^{1/2} 2\left(\tfrac{1}{2} - x\right)dx = \left(x - \tfrac{1}{2}\right)^2\Big]_{1/2}^{0} = \frac{1}{4}.$$

b). Find the marginal densities: f_X, f_Y.

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x), \quad f_X(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^y 2\,dx = 2y; \quad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

c) ~~Are~~ X and Y $\perp\!\!\!\perp$? $f_X(x) f_Y(y) = 4y(1-x) \neq 2 \quad \cancel{\forall} x, y$

X and Y are dependent.

W6-P41

$$f(x,y) = \begin{cases} c\,x^2 y & -y \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$-1 < \boxed{-x \leq y \leq 1}$

$0 < x < 1$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} \int_0^1 \frac{30}{7} x^2 y\,dy & 0 < x < 1 \\ \int_{-x}^1 \frac{30}{7} x^2 y\,dy & -1 \leq x \leq 0 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{15}{7} x^2 & 0 < x < 1 \\ \frac{15}{7} x^2 (1 - x^2) & -1 \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(x) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_{-y}^{1} \frac{30}{7} x^2 y\,dx = \begin{cases} \frac{10}{7}(y + y^4) & 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

solution:

$$P(X < 0) = \frac{2}{7} = F_X(0).$$

$$P(0 \leq Y \leq X \leq 1) = P(0 < Y < X < 1) = \frac{3}{7}$$

In general we have $f_{(X,Y)}(x,y) \neq f_X(x) f_Y(y)$:

Hence X and Y are dependent.

If X and Y were independent we would have

$$\frac{30}{7} x^2 y = \frac{15}{7} x^2 \times \frac{10}{7} y (1 + y^3) \quad \text{for } 0 \leq x, y \leq 1$$

$$2 = \frac{10}{7}(1 + y^3) \iff 0.4 = y^3; \quad \cancel{y^3 = \frac{1}{2^3}} \quad \cancel{y = \frac{1}{2}}$$

W6-P49

If we take $y = \frac{1}{2}$, $f_X(x)\, f_Y\left(\frac{1}{2}\right) = \frac{15}{7}x^2 \cdot \frac{10}{7}\left(\frac{1}{2} + \frac{1}{2^4}\right)$

$\boxed{0 < x < 1}$

$$\neq$$

$$f_{(X,Y)}\left(x, \frac{1}{2}\right) = \frac{30}{7}x^2 \cdot \frac{1}{2}$$

Hence X and Y are dependent.

$$P(X<0) = \int_{-\infty}^{0} f_X(x)\,dx = \int_{-1}^{0} \frac{15}{7}x^2(1-x^2)\,dx = \frac{2}{7}$$

$$P(0 \le Y \le X \le 1) = \iint_{\{0 \le y \le x \le 1\}} f(x,y)\,dx\,dy = \iint_A \frac{30}{7}x^2 y\,dx\,dy$$

where $A = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1,\ \cancel{y} \le x \le 1,\ 0 \le y \le 1\}$ ($0 \le x \le 1$).

$$\iint_A \frac{30}{7}x^2 y\,dx\,dy = \int_0^1 \left(\int_0^x \frac{30}{7}x^2 y\,dy\right) dx = 3/7$$

$$||$$

$$\int_0^1 \left(\int_y^1 \frac{30}{7}x^2 y\,dx\right) dy = \frac{3}{7}$$

$f_X(x) = 1$??!!

Be carefull

$$f(x,y) = \begin{cases} 1 & x \le y \le x+1,\ 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_x^{x+1} dy = 1$$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = E[X] = \int_0^1 x\,dx = \frac{1}{2}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_{?}^{?} 1\,dx$$

- $0 \leq x \leq 1$.

$0 \leq x \leq y \leq x+1$; $\quad y \geq x$ and $x \geq y-1$

$$\boxed{\begin{array}{l} x = 1 \quad 1 \leq y \leq 2 \\ x = 0 \quad 0 \leq y \leq 1 \end{array}}$$

$y-1 \leq x \leq y$

$\boxed{0 \leq y \leq 2}$ we have two cas.

1 case: $0 \leq y \leq 1 \implies 0 \leq x \leq y$.

$$f_Y(y) = \int_0^y 1\,dx = y.$$

2 case $1 \leq y \leq 2$. $0 \leq y-1 \leq x \leq y$ and $x \leq 1 \leq y$.
$\qquad\qquad\qquad\qquad\qquad\qquad\leq 1$

$$f_Y(y) = \int_{y-1}^{1} 1\,dx = 2-y \qquad 1 \leq y \leq 2$$

c/c

$$f_Y(y) = \begin{cases} y & \text{if } 0 \leq y \leq 1 \\ 2-y & \text{if } 1 < y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

~~$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = \int_0^1 y^2\,dy + \int_1^2 y \cdot 1 \cdot dy = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}$~~

σ_X, σ_Y, ρ ? = greek letter rho

W6-P44.

if "$f_{(X,Y)}(x,y) - f_X(x) f_Y(y)$" $= 0$ for all x and $y \in \mathbb{R}$
$\Rightarrow X \perp\!\!\!\perp Y$.

If $\exists\, x_0, y_0$, s.th $f(x_0, y_0) \neq f_X(x_0) f_Y(y_0)$ then X and Y are dependent.

for any $x, y \in [0,1]^2$

$$f_{X,Y}(x,y) - f_X(x) f_Y(y) = 1 - y \neq 0 \;\forall\, 0 < y < 1$$

$$\cancel{X\,and}\; f\left(0, \tfrac{1}{2}\right) - f_X(0) f_Y\left(\tfrac{1}{2}\right) = 1 - \tfrac{1}{2} = \tfrac{1}{2} \neq 0$$

then X and Y are dependent.

σ_X? $\quad E[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}. \Rightarrow \sigma_{\cancel{X}} = \sqrt{\frac{1}{3} - \frac{1}{2^2}}$

$$\sigma_X = \frac{1}{\sqrt{12}}; \quad \cancel{E[Y^2] = \int_0^1 y^3\,dy + \int_1^2 y^2\,dy = \frac{31}{12}}.$$

$$\cancel{\sigma_Y = \sqrt{\frac{31}{12} - \left(\frac{11}{6}\right)^2}} =$$

$$E(Y) = \int_0^1 y^2\,dy + \int_1^2 y(2-y)\,dy = 1$$

$$E(Y^2) = \frac{7}{6}. \Rightarrow \sigma_Y = \sqrt{\frac{1}{6}}.$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{7}{12} - \frac{1}{2} = \frac{1}{12}$$

Now $E[XY] = \int_0^1 \int_x^{x+1} xy\,dy\,dx = \frac{7}{12}$.

W6-P45

$\rho = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y} = \frac{1}{\sqrt{2}} \simeq 0.70$ .. $\boxed{Y = X + \frac{1}{2}}$

regression analysis.

$\Downarrow$

$\exists$ a linear correlation between X and Y.

$Y = \alpha X + \beta \Rightarrow$ taking expectation and variance in both sides we get equations in α and β which leads to: $\alpha = 1$, $\beta = \frac{1}{2}$;

Solutions of exercises:

P6. Q.1: $f(x,y) = \begin{cases} 4xy & 0<x<1 \text{ and } 0<y<1. \\ 0 & \text{otherwise:} \end{cases}$

The loss is $X+Y$. deductible 1.

- If $X+Y \leq 1$, then the insurer has to pay the loss him self. and the ~~loan~~ insurance company pays nothing.
- If $X+Y > 1$, then the insurance company pays $X+Y-1$ and the insurer pays 1.

Let us denote by Z the amount that will be paid by the insurance company if there is a claim ~~happens~~.

then $Z = \max(X+Y-1;0) = \begin{cases} X+Y-1, & \text{if } X+Y>1 \\ 0 & \text{if } X+Y\leq 1 \end{cases}$

We are asked to calculate $E[Z]$.

$$E[Z] = E[\max(X+Y-1;0)] = \int_0^1\int_0^1 \max(x+y-1;0)\,4xy\,dx\,dy.$$

$$= \int_0^1 \int_{\{x+y\geq 1,\ 0<y<1\}} (x+y-1)\,4xy\,dx\,dy + \int_0^1\int_{\{x+y<1,\ 0<y<1\}} 0\cdot 4xy\,dx\,dy$$

$$= \int_0^1\left(\int_{1-x}^{1}(x+y-1)\,4xy\,dy\right)dx \qquad 0<y<1. \quad 0<1-x<y<1$$

Then do the calculations. $\boxed{\cdot}$

$$\text{"}\max(x,0)\text{"} = \begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x\leq 0 \end{cases} \qquad \max(a,b) = \begin{cases} a & \text{if } a\geq b \\ b & \text{if } b\geq a. \end{cases}$$

W6-P47

$$\int_0^1 \left(\int_{1-y}^1 (x+y-1)\,4xy\,dx \right) dy \qquad \begin{matrix} 0 < x < 1 \\ x+y > 1 \\ (1-y < x < 1) \end{matrix}$$

$\boxed{-}$

$f_X(x)$? and $f_Y(y)$?

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) f_Y(y) = f(x,y) \Leftrightarrow X \perp\!\!\!\perp Y.$$

<u>P6 Q2</u>: $\mathbb{P}(X+Y \geq 1) = \iint_{\{x+y \geq 1\}} f(x,y)\,dx\,dy$

$$= \int_0^1 \left(\int_{1-x}^2 \frac{1}{4}(2x+2-y)\,dy \right) dx \qquad \begin{matrix} 0 < y < 2 \\ 1-x < y \end{matrix}$$

$\boxed{\infty}$

<u>P3 Q ii)</u> c.d.f of (X,Y).

$$F_{(X,Y)}(x,y) = \mathbb{P}(X \leq x, Y \leq y).$$

see solutions on the Bb.

P2. a) $X \hookrightarrow B(n+1, \frac{1}{2})$, $Y \hookrightarrow B(n, \frac{1}{2})$

$P(X-Y=k)$, $k \in \mathbb{Z}$ | $S_{X-Y} = \{$

$S_X = \{0, 1, \dots, n+1\}$, $S_Y = \{0, \dots, n\}$

$Z = X - Y$ is also a r.v. Find distribution p.m.f of Z.

$S_Z = \{-n, -n+1, \dots -1, 0, 1, 2, \dots, n+1\}$. $\boxed{k \in S_Z}$

$$P(X-Y=k) = P(X = Y+k) \qquad \Omega = \bigcup_{i=0}^{n} \{Y=i\}$$

$$= \sum_{i=0}^{n} P(X = Y+k, Y=i)$$

$$= \sum_{i=0}^{n} P(X = i+k, Y=i) = \sum_{i=0}^{n} P(X=i+k) P(Y=i)$$

$$= \sum_{i=0}^{n} \binom{n+1}{i+k} \frac{1}{2^{n+1}} \binom{n}{i} \frac{1}{2^n} = \frac{1}{2^{2n+1}} \sum_{i=0}^{n} \binom{n+1}{i+k}\binom{n}{i}$$

If X r.v. with sample space S_X

$$\Omega = \bigcup_{x \in S_x} \{X = x\}, \quad P(\Omega) = 1.$$

$P(X > Y)$, $P(X = 6Y)$

$$= \sum_{x \in S_X} P(Y \leq x, X = x) \qquad \sum_{y \in S_Y} P(X = 6y, Y = y) = P(X=6y)P(Y=y)$$

If $X \perp\!\!\!\perp Y$.

$$\sum_{x \in S_x} P(Y \leq x) P(X = x)$$

W6-P49

$P(X=m, Y=n) = \frac{e^{-1}}{n!}\frac{1}{2^m}$. $m \geq 1$, $n \geq 0$.

Distribution of X. • $S_X = \{1, 2, \dots\} = \mathbb{N}^*$.

$m \geq 1$, pmf of X: $P(X=m) = \sum_{n=0}^{\infty} P(X=m, Y=n)$.

$\{X=m\} = \{X=m\} \cap \Omega = \{X=m\} \cap \left(\bigcup_{n=0}^{\infty}\{Y=n\}\right)$.

$= \bigcup_{n=0}^{\infty} \{X=m, Y=n\}$.

$$P(X=m) = \sum_{n=0}^{\infty} \frac{1}{2^m}\frac{e^{-1}}{n!} = \frac{1}{2^m} e^{-1}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\right) = \frac{1}{2^m} e^{-1} e^{1}.$$

$X \hookrightarrow G\left(\frac{1}{2}\right)$.

similarly: $Y \hookrightarrow \mathcal{P}(1)$.

• $X \perp\!\!\!\perp Y$ because $P(X=m, Y=n) = P(X=m)P(Y=n)$.

d) X, Z r.v. $Z \hookrightarrow \mathcal{P}(\lambda)$. $X \leq Z$.

$P(X=k \mid Z=n) = \binom{n}{k} p^k (1-p)^{n-k}$.

"$Y = Z - X$"

X and Y are $\perp\!\!\!\perp$ Poisson. first Distribution of X.

① $S_X = \mathbb{N}$ ② pmf of X: $P(X=k) = P\left(X=k, \bigcup_{j=0}^{\infty}\{Y=j\}\right)$

• $S_Y = \mathbb{N}$;

$= \sum_{j=0}^{\infty} P(X=k, Y=j)$

$= \sum_{j=0}^{\infty} P(X=k, Z-X=j)$

N6 P50

$$P(X=k) = \sum_{j=0}^{\infty} P(X=k, Z=j+k).$$

$$= \sum_{j=0}^{\infty} P(X=k \mid Z=j+k) P(Z=j+k).$$

$$= \sum_{j=0}^{\infty} \binom{j+k}{k} p^k (1-p)^j e^{-\lambda} \frac{\lambda^{j+k}}{(j+k)!}$$

$$= \sum_{j=0}^{\infty} \frac{\cancel{(j+k)!}}{k!\, j!} p^k (1-p)^j \frac{\lambda^{j+k}}{\cancel{(j+k)!}}$$

$$= \frac{(p\lambda)^k e^{-\lambda}}{k!} \left(\sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!} \right) = \frac{(p\lambda)^k e^{-\lambda}}{k!} e^{\lambda(1-p)}.$$

$$= \frac{e^{-\lambda p} (\lambda p)^k}{k!} \Longrightarrow X \hookrightarrow P(p\lambda).$$

W6-PS1