HW3 P2 (a) i) $\mathbb{P}(X=Y)$??

$$\{X=Y\} = \{X=Y\}\cap\Omega, \quad \Omega = \bigcup_{j=0}^{n}\{Y=j\}.$$

$$= \{X=Y\}\cap\left(\bigcup_{j=0}^{n}\{Y=j\}\right).$$

$$= \bigcup_{j=0}^{n}\left(\{X=Y\}\cap\{Y=j\}\right)$$

$$= \bigcup_{j=0}^{n}\{X=j; Y=j\}.$$

$$\mathbb{P}(X=Y) = \sum_{j=0}^{n}\mathbb{P}(X=j; Y=j) = \sum_{j=0}^{n} P(X=j)P(Y=j) \quad (X\perp\!\!\!\perp Y)$$

$$= \sum_{j=0}^{n}\binom{n+1}{j}\left(\frac{1}{2}\right)^{j}\left(\frac{1}{2}\right)^{n+1-j}\binom{n}{j}\left(\frac{1}{2}\right)^{j}\left(\frac{1}{2}\right)^{n-j}$$

$$= \sum_{j=0}^{n}\binom{n+1}{j}\binom{n}{j}\underbrace{\frac{1}{2^{n+1}}\cdot\frac{1}{2^{n}}} = \frac{1}{2^{2n+1}}\sum_{j=0}^{n}\binom{n+1}{j}\binom{n}{j}$$

$$\{X>Y\} = \{X>Y\}\cap\Omega = \{X>Y\}\cap\left(\bigcup_{j=0}^{n}\{Y=j\}\right)$$

$$= \bigcup_{j=0}^{n}\{X>j; Y=j\}.$$

$$P(X>Y) = \sum_{j=0}^{n}\mathbb{P}(X>j; Y=j) = \sum_{j=0}^{n} P(X>j)\mathbb{P}(Y=j).$$

$$= \sum_{j=0}^{n}\binom{n}{j}\frac{1}{2^{n}}\left(\sum_{k=j+1}^{n+1} P(X=k)\right)$$

$$= \sum_{j=0}^{n}\binom{n}{j}\frac{1}{2^{n}}\sum_{k=j+1}^{n+1}\binom{n+1}{k}\frac{1}{2^{n+1}}$$

$$= \frac{1}{2^{2n+1}}\sum_{j=0}^{n}\binom{n}{j}\cdot\sum_{k=j+1}^{n+1}\binom{n+1}{k}. \quad \boxed{\cdot}$$

X, Y $P(X\leq Y+4)$
$\mathbb{P}(X\geq 5Y)$
$\mathbb{P}(Y=4X)$

W7-P52

HW1 P4. Q4. a): Find m such that $\boxed{mC = 120}$

$P(X > m) \leq 1\% \iff P(X \leq m) > 99\%$.

$$\sum_{k=0}^{m} \frac{20!}{k!\,(20-k)!} \left(\frac{2}{100}\right)^k \left(\frac{98}{100}\right)^{20-k}$$

$m = 0 \rightarrow \left(\frac{98}{100}\right)^{20} < 99\%$

$m = 1 \rightarrow 0.9401$

$m = 2 \rightarrow 0.99293 > 99\%$

$\underline{m = 1}$ $\left(\frac{98}{100}\right)^{20} + 20\,\frac{2}{100}\cdot\left(\frac{98}{100}\right)^{19} = 0.9401$.

$m = 2 \Rightarrow 2C = 120 \Rightarrow C = 60$.

HW3. P2 (b): X_1 and X_2 i.i.d. & $S_{X_1} = S_{X_2} = \mathbb{N}$.

pmf $f(k) = P(X_1 = k) = P(X_2 = k) = \frac{1}{2^{k+1}}$, $k \geq 0$

$X_1 \hookrightarrow G\left(\frac{1}{2}\right)$, $X_2 \hookrightarrow G\left(\frac{1}{2}\right)$.

$Y = \max(X_1, X_2)$.

- $S_Y = \mathbb{N}$, $n \in \mathbb{N}$, pmf. $f(n) = P(Y = n)$?

$\{Y = n\} = \{\max(X_1, X_2) = n\} = \cancel{\{\max(X_1, X_2) = n\} \cap \{X_1 < X_2\}}$

$= \{\max(X_1, X_2) = n\} \cap \left[\{X_1 < X_2\} \cup \{X_2 \leq X_1\}\right]$

$$\max(a, b) = \begin{cases} a & a > b \\ b & b \geq a \end{cases}$$

$= \{\max(X_1, X_2) = n\} \cap \{X_1 < X_2\}$

$\cup \{\max(X_1, X_2) = n\} \cap \{X_2 \leq X_1\}$.

$$\boxed{\{Y = n\} = \{X_2 = n,\ X_1 < \underset{\overset{\parallel}{n}}{X_2}\} \cup \{X_1 = n,\ X_2 \leq \underset{\overset{\parallel}{n}}{X_1}\}.}$$

W7-P53

Chapter 3: Conditional expectation:

1. Conditioning on an event:

Let $(\Omega, \mathcal{F}, P)$ be a probability space and B an event, that is $B \in \mathcal{F}$, such that $P(B) > 0$.

Let X be a random variable such that $E|X| < +\infty$.

Def: The conditional expectation of X given B is defined by $E[X|B] = \frac{1}{P(B)} E[X \mathbb{1}_B]$.

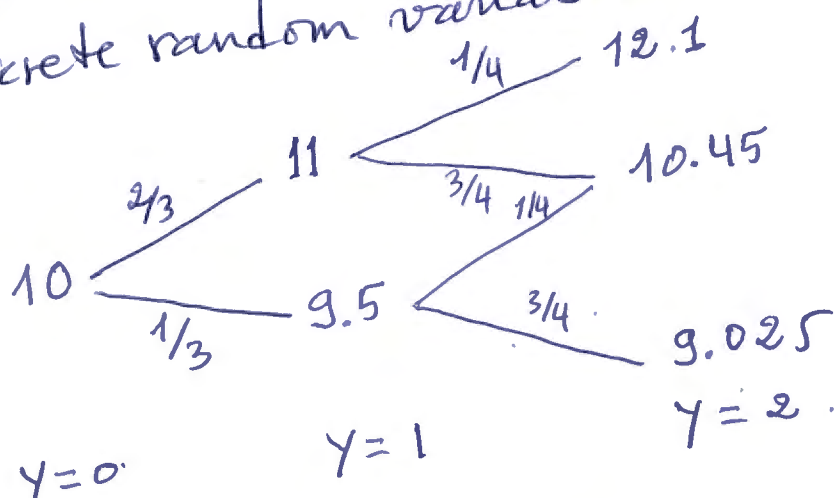
where $\mathbb{1}_B = \begin{cases} 1 & \text{on } B \\ 0 & \text{on } B^c. \end{cases}$

Remark: We can also write $E[X|B]$ using conditional distribution:

$$E[X|B] = \sum_{x \in S_X} x P(X = x|B)$$

if X is a discrete random variable.

Example:



Let X be the values of the year 2.

$S_X = \{9.025; 10.45; 12.1\}$ with pmf $\frac{1}{4}; \frac{7}{12}; \frac{1}{6}$.

We denote by u the event price rises and d event price moves down. Set $B = \{u\}$.

W7-P54

Question find $E[X \mid B]$?

We have $E[X \mid B] = \sum_{x \in S_X} x P(X = x \mid B)$.

which can be written also as follows $= \sum_{x \in S_X} x \dfrac{P(X=x, B)}{P(B)}$.

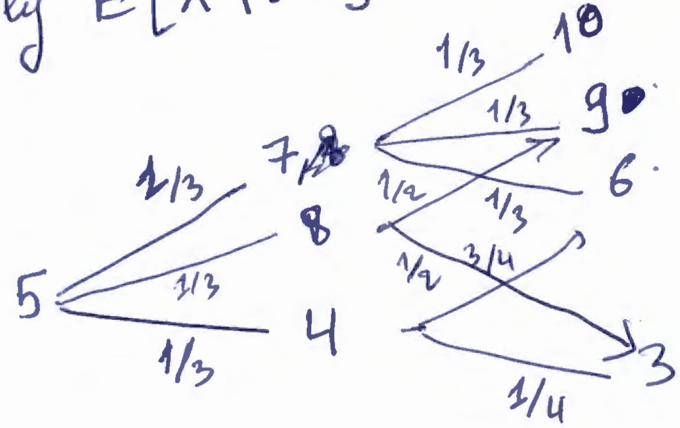
Hence: $E[X \mid \{u\}] = 12.1\, P(X = 12.1 \mid \{u\}) + 10.45\, P(X = 10.45 \mid \{u\}) + 9.025\, P(X = 9.025 \mid \{u\})$

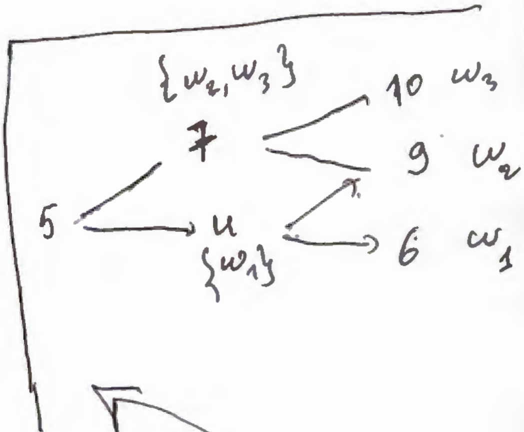
Remark first that $P(X = 9.025 \mid \{u\}) = 0$.

therefore. $E[X \mid \{u\}] = 12.1 \times \frac{1}{4} + 10.45 \times \frac{3}{4} = 10.8625$

Similarly $E[X \mid \{d\}] = 10.45 \times \frac{1}{4} + 9.025 \times \frac{3}{4} = 9.3813$.

Example:





~~$B = \{u$~~ Consider an other example.

$$\boxed{S_0 = 5, \quad S_1(w_1) = 4, \quad S_1(w_2) = S_1(w_3) = 7 \quad S_2(w_1) = 6, \quad S_2(w_2) = 9 \text{ and } S_2(w_3) = 10}$$

Denote by. $q_i = P(\{w_i\})$. $i = 1, 2, 3$.

Set $B = \{w_2, w_3\}$. Find $E[S_2 \mid B]$?

do it as an exercise at home. | $\Omega = \{w_1, w_2, w_3\}$.

If X is a discrete r.v. and B an event such that $P(B) > 0$: The conditional p.m.f of X given the event B is given by:

$$f(x|B) = P(X=x|B) = \frac{P(X=x, B)}{P(B)}.$$

and the conditional expectation of X given B is:

$$E[X|B] = \sum_{x \in S_X} f(x|B).$$

If $(B_n)_{n \geq 0}$ is a partition of the sample space Ω, then $E[X] = \sum_{n \geq 0} E[X|B_n] P(B_n)$.

2- Conditioning on a discrete random variable.

Given an arbitrary random variable X such that $E|X| < +\infty$ and a discrete r.v. $Y: \Omega \to \{y_1, \dots, y_m\}$
The conditional expectation of X given Y:

$E[X|Y]$ must depend solely on the r.v. Y.

Thus $E[X|Y]$ is itself a r.v.

Def The conditional expectation of X given Y is defined to be a discrete r.v. $E[X|Y]: \Omega \to \mathbb{R}$.
such that: $E[X|Y](\omega) = E[X|Y=y_i]$ on the set $\{\omega : Y(\omega) = y_i\}$, $i = 1, 2, \dots, m$.

W7.PTA

Example: Three coins, 10, 20 and 50 cents are tossed. The values of those coins ~~are~~ with heads up are added for a total amount X.

Let Y be the total of the 10 and 20 cents.

Find $E[X|Y]$. $S_Y = \{0, 10, 20, 30\}$.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

the corresponding probabily is uniform $= \frac{1}{8}$.

$$E[X|Y=0] = \sum_{x \in S_X} x P(X=x|Y=0).$$

$$= 10 P(X=10|Y=0) + 20 P(X=20|Y=0) + 30 P(X=30|Y=0) + 50 P(X=50|Y=0) + 60 P(X=60|Y=0) + 70 P(X=70|Y=0) + 80 P(X=80|Y=0).$$

$$= 50 P(X=50|Y=0) = P(TTH|TT) \times 50$$

$$= \frac{50}{2} = 25.$$

$$E[X|Y=10] = 10 P(X=10|Y=10) + 60 P(X=60|Y=10)$$
$$= 10 P(HTT|HT) + 60 P(HTH|HT).$$
$$= \frac{10}{2} + \frac{60}{2} = 35.$$

$$E(X|Y=20] = 20 P(X=20|Y=20) + 70 P(X=70|Y=20)$$
$$= 20 P(THT|THT) + 70 P(THH|TH).$$
$$= \frac{20}{2} + \frac{70}{2} = 45$$

$$E(X|Y=30] = 30 P(X=30|Y=30] + 80 P(X=80|Y=30).$$
$$= 30 P(HHT|HH) + P(HHH|HH).80$$
$$= \frac{30}{2} + \frac{80}{2} = 55.$$

x07.957

$$E[S_2 \mid B] = \frac{1}{P(B)} E[S_2 1_B] = \frac{1}{P(B)} \int_B X \, dP$$

$$= \sum_{x \in S_{S_2}} x P(S_2 = x \mid B)$$

$$= 6 P(S_2 = 6 \mid B) + 9 P(S_2 = 9 \mid B) + 10 P(S_2 = 10 \mid B)$$

$$= 6 \underbrace{P(\{\omega_1\} \mid \{\omega_2, \omega_3\})}_{=0} + 9 P(\{\omega_2\} \mid \{\omega_2, \omega_3\}) + 10 P(\{\omega_3\} \mid \{\omega_2 \omega_3\})$$

$$= 9 \frac{q_2}{q_2 + q_3} + 10 \frac{q_3}{q_2 + q_3}.$$

$$P(\{\omega_1\} \mid \{\omega_2, \omega_3\}) = \frac{P(\{\omega_1\} \cap \{\omega_2, \omega_3\})}{P(\{\omega_2; \omega_3\})} = \frac{P(\{\phi\})}{q_2 + q_3} = 0$$

$$P(\{\omega_2, \omega_3\}) = P(\omega_2) + P(\omega_3) = q_2 + q_3.$$

$Z = E[X \mid Y]$; $S_Y = \{0, 10, 20, 30\}$.

$$S_Z = \{ \underbrace{E[X \mid Y=0]}_{= z_1}, \underbrace{E[X \mid Y=10]}_{= z_2}, \underbrace{E[X \mid Y=20]}_{= z_3}, \underbrace{E[X \mid Y=30]}_{= z_4} \}$$

See the page W7-P57 (previous page).

$$\text{c/c: } E[X \mid Y] = \begin{cases} 25 & \{Y=0\} \quad TT \\ 35 & \{Y=10\} \quad HT \\ 45 & \{Y=20\} \quad TH \\ 55 & \{Y=30\} \quad HH \end{cases}$$