Let X and Y be two r.v. with joint pmf or pdf:

$f(x,y)$, $x \in S_X$, $y \in S_y$:

$E[X|Y]$ is a r.v. which can be written as a function of Y. That is $\exists g$: such that:

$$E[X|Y] = g(Y).$$

The set of all possible values of $E[X|Y]$ is given by $\{g(y), y \in S_y\}$, where:

$$g(y) = E[X|Y=y] = \begin{cases} \sum_{x \in S_X} x f(x|y) \\ \int_{S_X} x f(x|y)\,dx. \end{cases}$$

$$f(x|y) = \frac{f(x,y)}{f(y)}, \quad f(y) = \int_{S_X} f(x,y)\,dx \text{ or } \sum_{x \in S_X} f(x,y).$$

$$E(h(Y)/X) = \varphi(X), \text{ where:}$$

$$\varphi(x) = E[h(Y)|X=x] = \begin{cases} \sum_{y \in S_y} h(y) f(y|x). \\ \int_{S_y} h(y) f(y|x)\,dy \end{cases}$$

So If we know joint pdf we can find the conditional expectation of X given Y.

Example: Roll a die until we get a 6.

Let Y be the number of rolls and X the number of 1's we get.

Find $E[X|Y]$?

$$S_Y = \{1, 2, \dots, \} = \mathbb{N}^*. \quad Y \hookrightarrow \mathcal{G}\left(\frac{1}{6}\right)$$

$$n \geq 1 \quad P(Y=n) = P(\text{The } (n-1) \text{ rolls were not 6 and the } n^{th} \text{ roll we get } 6).$$

$$= P((n-1) \text{ rolls with out } 6)\, P(n^{th}\ 6).$$

$$= \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} = \underbrace{\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdots\frac{5}{6}}_{n-1}\cdot\frac{1}{6}$$

$E[X|Y]$ is a r.v. with values: $E[X|Y=n]$ $n \geq 1$.

Now we need to compute $E(X|Y=n)$.

We have

$$E[X|Y=n] = \sum_{i=0}^{n-1} i\, P(X=i|Y=n).$$

$$P(X=i|Y=n) = \binom{n-1}{i}\left(\frac{1}{5}\right)^{i}\left(\frac{4}{5}\right)^{n-1-i}.$$

$$E[X|Y=n] = \sum_{i=0}^{n-1} i\binom{n-1}{i}\left(\frac{1}{5}\right)^{i}\left(\frac{4}{5}\right)^{n-1-i}$$

$$\cancel{= \sum_{i=1}^{n-1} \frac{(n-1)!}{(n-1-i)!\,(i-1)!}\,\frac{1}{6}\left(\frac{5}{6}\right)^{n-1-i}}$$

$$\cancel{\sum_{i=1}^{n-1}}$$

W8·P60

$$E[X|Y=n] = \sum_{k=0}^{n-1} k \binom{n-1}{k} \frac{1}{5^k}\left(\frac{4}{5}\right)^{n-1-k}.$$

$$\cancel{= \sum_{k=0}^{n-1} \frac{n(n-1)!}{n\,k!\,(n-1-k)!}\frac{1}{5^k}\left(\frac{4}{5}\right)^{n-1-k}}$$

$$\cancel{= \frac{1}{n}\sum_{k=0}^{n-1} \frac{n!}{(n-(k+1))!\,k!}\frac{1}{5^k}\left(\frac{4}{5}\right)^{n-1-k}}$$

$\cancel{n=5}$

$$E[X|Y=5]$$

$$= (n-1)\cdot\frac{1}{5} = \frac{n-1}{5}.$$

$$E[X|Y] = \frac{Y-1}{5} = g(Y),\quad g(n) = \frac{n-1}{5}.$$

Properties of conditional expectation:

- $E[\alpha_1 X_1 + \alpha_2 X_2 | Y] = \alpha_1 E[X_1|Y] + \alpha_2 E[X_2|Y]$
- $E[X|Y] = E[X]$ when $X \perp\!\!\!\perp Y$.
- $E[E[X|Y]] = E[X]$.
- $E[E[Y|X]] = E[Y]$.
- $Var(X|Y=y) = Var(X)$ if $X \perp\!\!\!\perp Y$.
- $Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$ | If $X \perp\!\!\!\perp Y$ then $X^2 \perp\!\!\!\perp Y$

$$= E[X^2] - (E[X])^2$$

$$= Var(X).$$

let $X_1, X_2, \dots, X_n$ be a family of r.v.

$$E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} E[X_k].$$

$$Var\left(\sum_{k=1}^{n} X_k\right) = Covar\left(\sum_{k=1}^{n} X_k, \sum_{k=1}^{n} X_k\right) \text{ because}$$

$$Var(X) = cov(X, X)$$

Then $$Var\left(\sum_{k=1}^{n} X_k\right) = Cov\left(\sum_{k=1}^{n} X_k, \sum_{j=1}^{n} X_j\right)$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n} cov(X_k, X_j)$$

Because $$Cov(X, Y+Z) = E[X(Y+Z)] - E[X]E[Y+Z] - E[X]E[Z].$$

$$= E[XY] + E[XZ] - E[X]E[Y] - E[X]E[Z]$$

$$Cov(X, Y+Z) = cov(X,Y) + Cov(X,Z).$$

similarly we have.

$$cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z).$$

Therefore: $$Cov\Big(\underbrace{\sum_{k=1}^{n} X_k}_{X}, \underbrace{\sum_{k=1}^{n} X_k}_{X_1+Z}\Big) = Cov\Big(\underbrace{\sum_{k=1}^{n} X_k}_{X}, \underbrace{X_1}_{Y} + \underbrace{\sum_{k=2}^{n} X_k}_{Z}\Big)$$

$$= Cov\left(\sum_{k=1}^{n} X_k, X_1\right) + Cov\left(\sum_{k=1}^{n} X_k, \sum_{k=2}^{n} X_k\right)$$

$$= \sum_{j=1}^{n} Cov\left(\sum_{k=1}^{n} X_k, X_j\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} Cov(X_k, X_j).$$

W8P62

If $(X_k)_{k\geq 1}$ are independent then

$$\mathrm{Cov}\left(\sum_{k=1}^{n} X_k, \sum_{j=1}^{n} X_j\right) = \sum_{k=1}^{n} \mathrm{Cov}(X_k, X_k).$$

because if $X_k \perp\!\!\!\perp X_j$ $\mathrm{cov}(X_k, X_j) = 0 \;\forall k \neq j$.

That is $\mathrm{Var}\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \mathrm{Var}(X_k)$. $(X_k \perp\!\!\!\perp X_j)\, \forall_{k\neq j}$

Let X and Y be ~~two~~ two random variables.

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Cov}(X+Y, X+Y) \\ &= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y) \\ &= \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y).\end{aligned}$$

Remark If $\mathrm{Cov}(X,Y) = 0 \Rightarrow$ does not implies in general that $X \perp\!\!\!\perp Y$.

If $X \perp\!\!\!\perp Y \Longrightarrow \mathrm{Cov}(X,Y) = 0$.

But $\mathrm{Cov}(X,Y) = 0 \not\Longrightarrow X \perp\!\!\!\perp Y$ (in general).

Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. random variables. Then $E[X_1] = E[X_2] = E[X_n] = \mu$ and $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \ldots \mathrm{Var}(X_n) = \sigma^2 \;\forall n \geq 1$

Consider N a Poisson r.v. with parameter $\lambda > 0$ such that N is $\perp\!\!\!\perp (X_1, X_2, X_3, \ldots, X_n, \ldots)$

WB-263

Set $X = \sum_{k=1}^{N} X_k = X_1 + X_2 + \cdots + X_N$ fog.

This r.v X is used as a model loss payment of an insurance company. The insurance company wants to know the expected loss for some period and also its standard deviation:

$$E[X]? \text{ and } \sqrt{Var(X)} = sd(X) = \sigma_X ?$$

The distribution of X is not known.

First $E[X|N]$? and $Var(X|N)$?

$$E[X|N] = h(N) \text{ where } h(n) = E[X|N=n].$$

Then $h(n) = E\left[\sum_{k=1}^{N} X_k \middle| N=n\right] = E\left[\sum_{k=1}^{n} X_k \middle| N=n\right]$.

$$= \sum_{k=1}^{n} E[X_k | N=n] = \sum_{k=1}^{n} E[X_k]$$

$$= \sum_{k=1}^{n} \mu = \mu n$$

Therefore $E[X|N] = h(N) = \mu N$.

$$E[X] = E[E[X|N]] = E[\mu N] = \mu E[N]$$
$$= \mu \lambda.$$

$\boxed{\text{Wednesday 2 or 3 exercises.}}$