King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1442
M 507 - Advanced Operations Research
Time: 3 hours

Answer the following questions:

Q1: [3+2+4]

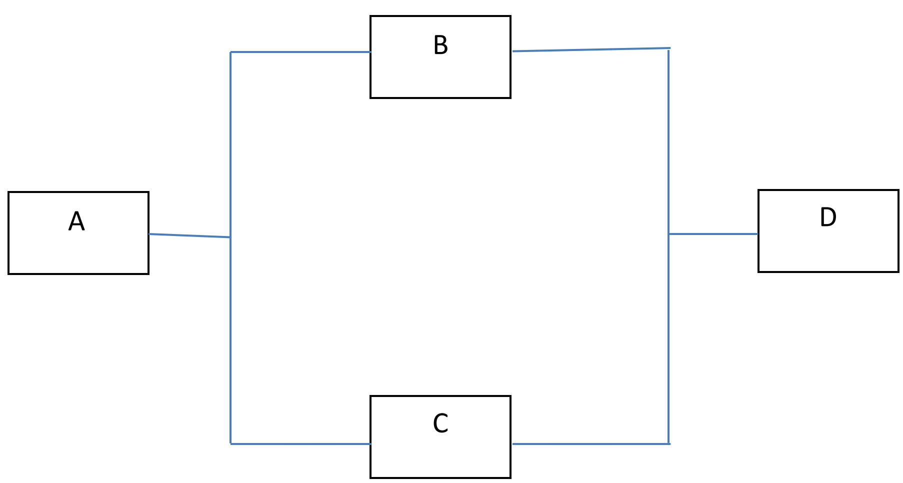
In the reliability diagram below, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

a. Decomposition using B as the keystone element.

b. The reduction method.

c. Compute the importance of each component if $R_A = 0.85$, $R_B = 0.95$,

$R_C = 0.9$ and $R_D = 0.98$



Q2: [5+2]

(a) If $X(t)$ represents a size of a population where $X(0) = 1$, using the following differential equations:

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\ (p), \;\; p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over [0,1]. Show that $pr\{X(U) = k\} = p^k / (\beta k)$ for $k = 1, 2, \ldots$, with $p = 1 - e^{-\beta}$.

Q3: [3+5]

(a) If a random variable $T \sim \text{Weibull}(\eta, \beta)$. Compute its failure rate function and MTTF.

(b) Suppose the life distribution of an item has failure rate function $\lambda(t) = t^3, \quad 0 < t < \infty$.
i) What is the probability the item survives to age 2?
ii) What is the probability that the item's life is between 0.4 and 1.4?
iii) What is the mean life of the item?

Q4: [4+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \text{pr}\{X_0 = i_0\}$

(b) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

ii) Determine the probability that a correct signal is received at stage 2.

Q5: [8]

A furniture company produces chairs and tables, subject to the board-foot and man-hour restrictions, our problem is to maximize the profit function $z = \$45x_1 + \$80x_2$ subject to

$$5x_1 + 20x_2 \leq 400$$
$$10x_1 + 15x_2 \leq 450$$
$$x_1 \geq 0 \;,\; x_2 \geq 0$$

Here, x_1 stands for the number of chairs and x_2 for the number of tables to be manufactured, where, we have a total of 400 board-feet of wood and 450 man-hours to combine into a manufacturing schedule for chairs and tables.

Model Answer

Q1: [3+2+4]

In the reliability diagram shown in Fig. 1, the reliability of each component is constant and independent. Assuming that each has the same reliability *R*, compute the system reliability as a function of *R* using the following methods:

a) Decomposition using B as the keystone element.

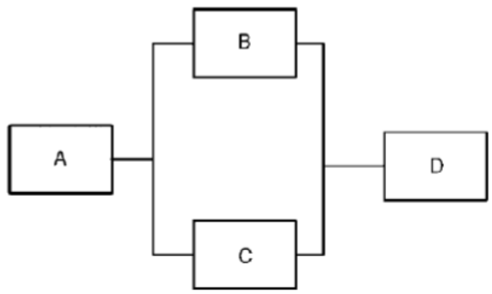


Fig. 1: Reliability diagram

Using B as the keystone element, we have two cases i.e., the case when B functions and the case when it does not.

For the case when B functions, the system reduced to Fig 2.

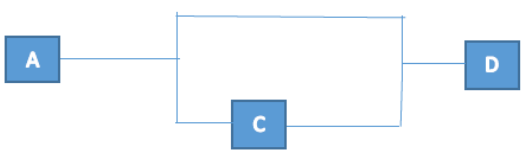


Fig. 2: The case when B functions

Thus the reliability of the system depends only on the reliability of component A and D. Note that $R_A = R_B = R_C = R_D = R$

Therefore,

$$R^+ = R_A R_D = R^2$$

For the case when B fails, the system block is as shown in Fig. 3, which is a series system.

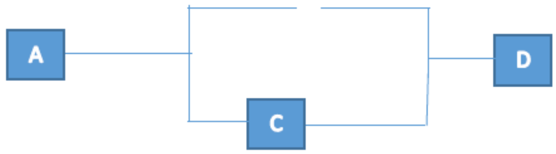


Fig. 3: The case when B fails to work

Thus the reliability of the system depends on A, C, and D, therefore we have:

$$R^{-} = R_A R_C R_D = R^3$$

Thus the reliability of the system using the two decompositions is given as:

$$R_{system} = R_B R^{+} + (1 - R_B) R^{-}$$
$$R_{system} = R(R^2) + (1 - R) R^3$$
$$R_{system} = 2R^3 - R^4$$

b) Using the reduction method

With this method, it can be seen that components B and C are in parallel and jointly in series with A and D. therefore the reduced system is given in Fig. 4.

Fig. 4: Reduced system

For parallel components B and C, we have

$$R_{B\|C} = 1 - \prod_{i=1}^{2} (1 - R_i)$$
$$R_{B\|C} = R_B + R_C - R_B R_C$$
$$R_{B\|C} = 2R - R^2$$

The reliability of the system is thus given as:

$$R_{system} = R_A R_{B\|C} R_D$$
$$R_{system} = R(2R - R^2) R$$
$$R_{system} = 2R^3 - R^4$$

c)

Recall that the reliability of the system is given as:

$$R_{system} = R_A R_D (R_B + R_C - R_B R_C)$$

The importance of each component is computed by taking the partial derivative with respect to each of the component.

Thus the importance of component A is given as:

$$\frac{\delta R_{system}}{\delta R_A} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_A}$$
$$I_A = R_D (R_B + R_C - R_B R_C)$$

The importance of component B is given as:

$$\frac{\delta R_{system}}{\delta R_B} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_B}$$
$$I_B = R_A R_D - R_A R_D R_C$$

The importance of component C is given as:

$$\frac{\delta R_{system}}{\delta R_C} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_C}$$
$$I_C = R_A R_D - R_A R_B R_D$$

The importance of component D is given as:

$$\frac{\delta R_{system}}{\delta R_D} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_D}$$
$$I_D = R_A (R_B + R_C - R_B R_C)$$

$\Rightarrow I_A = 0.9751,\ I_B = 0.0833,\ I_c = 0.04165$ and $I_D = 0.84575$

Q2: [5+2]

(a)

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t),\ n=1,2,3,\ \ldots \qquad (2)$$

The initial condition is $X(0) = 1 \quad \Rightarrow \quad p_1(0) = 1$

$$\Rightarrow\ p_n(0)=\begin{cases}1 & ,\ \text{n=1}\\ 0 & ,\ \text{otherwise}\end{cases}$$

$$\lambda_0=0 \qquad (1)\Rightarrow \frac{dp_0(t)}{dt}=0$$

$$\Rightarrow\ p_0(t)=0 \qquad (3)$$

$$(2)\Rightarrow\ \frac{dp_n(t)}{dt}=\lambda_{n-1}p_{n-1}(t)-\lambda_n p_n(t)$$

$$\Rightarrow\ \frac{dp_n(t)}{dt}+\lambda_n p_n(t)=\lambda_{n-1}p_{n-1}(t),\ \ n=1,2,\ \ldots$$

$$\because\ \lambda_n=n\lambda,\ \ \lambda_{n-1}=(n-1)\lambda$$

$$\therefore\ \frac{dp_n(t)}{dt}+n\lambda p_n(t)=(n-1)\lambda p_{n-1}(t),\ \text{n=1,2, ...}$$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt}+n\lambda p_n(t)\right]=(n-1)\lambda p_{n-1}(t)e^{n\lambda t}$$

$$\therefore\ \frac{d}{dt}\left[p_n(t)e^{n\lambda t}\right]=(n-1)\lambda p_{n-1}(t)e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x)e^{n\lambda x}\right]=(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx$$

$$\therefore\ \left[p_n(x)e^{n\lambda x}\right]_0^t=(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx$$

$$\Rightarrow\ p_n(t)=e^{-n\lambda t}\left[p_n(0)+(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx\right],\ \ n=1,2,\ \ldots \quad (4)$$

which is a recurrence relation.

at $n=1$

$$p_1(t)=e^{-\lambda t}\left[p_1(0)+0\right]=e^{-\lambda t} \qquad (5)$$

at $n=2$

$$p_2(t)=e^{-2\lambda t}\left[p_2(0)+\lambda\int_0^t p_1(x)e^{2\lambda x}dx\right]$$

$(5) \Rightarrow \; p_1(x) = e^{-\lambda x}$

$\therefore \; p_2(t) = e^{-2\lambda t}\left[\lambda \int_0^t e^{-\lambda x} e^{2\lambda x} dx\right]$

$\therefore \; p_2(t) = \lambda e^{-2\lambda t} \int_0^t e^{\lambda x} dx$

$= e^{-\lambda t}(1 - e^{-\lambda t})^1 \qquad (6)$

Similarly as (5) and (6), we deduce that

$p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$

$= p(1-p)^{n-1}, \;\; p = e^{-\lambda t}, \; n = 1, 2, \; ...$

$\therefore X(t) \sim geom(p), \;\; p = e^{-\lambda t}$

$Mean\left[X(t)\right] = 1/p = e^{\lambda t},$

$Variance\left[X(t)\right] = \dfrac{1-p}{p^2} = \dfrac{1 - e^{-\lambda t}}{e^{-2\lambda t}}$

(b) For Yule process,

$p_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \quad n \ge 1$

$\Rightarrow$

$\because pr\left\{X(U) = k\right\} = \int_0^1 e^{-\beta u}(1 - e^{-\beta u})^{k-1} du$

$= \dfrac{1}{\beta}\int_0^1 (1 - e^{-\beta u})^{k-1}. \; \beta e^{-\beta u} du$

$= \dfrac{1}{\beta}\left[\dfrac{(1 - e^{-\beta u})^k}{k}\right]_0^1$

$= \dfrac{1}{\beta k}\left[(1 - e^{-\beta})^k\right]$

$\therefore pr\left\{X(U) = k\right\} = \dfrac{p^k}{\beta k}, \; k = 1, 2, \; ...$ where $p = 1 - e^{-\beta}$

Q3: [3+5]

(a)

For $T \sim \text{Weibull}(\eta, \beta)$, the p.d.f. is

$f(t)=\frac{\beta}{\eta}\left[\frac{t}{\eta}\right]^{\beta-1}\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$ and the Reliability function is $R(t)=\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$

$$\because \lambda(t)=\frac{f(t)}{R(t)}$$

$$\therefore\ \lambda(t)=\frac{\beta}{\eta}\left[\frac{t}{\eta}\right]^{\beta-1} \qquad (1)$$

$$\because\ f(t)=\frac{\beta}{\eta}\left[\frac{t}{\eta}\right]^{\beta-1}\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

Let, $\alpha=\left(\frac{1}{\eta}\right)^{\beta}=\eta^{-\beta}$

$$\Rightarrow\ f(t)=\alpha\beta\ t^{\beta-1}e^{-\alpha t^{\beta}}$$

The Mean, $\mu=\int_0^{\infty}t\ \alpha\beta\ t^{\beta-1}e^{-\alpha t^{\beta}}dt$

Let, $u=\alpha t^{\beta}$

$$\Rightarrow\ \text{du}=\alpha\beta t^{\beta-1}dt,\ t=\left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}}$$

The Mean, $\mu=\int_0^{\infty}\left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}}e^{-u}du$

$$\mu=\alpha^{\frac{-1}{\beta}}\int_0^{\infty}(u)^{\frac{1}{\beta}}e^{-u}du$$

$$\mu=\alpha^{\frac{-1}{\beta}}\Gamma(1+\frac{1}{\beta})$$

$$\therefore\ \mu=\eta\Gamma(1+\frac{1}{\beta}),\ \eta=\alpha^{\frac{-1}{\beta}} \qquad (2)$$

(b)

i)

$$R(t)=\exp\left[-\int_0^t\lambda(x)dx\right]$$

$$R(2) = \exp\left[-\int_0^2 t^3 dt\right]$$

$$= \exp\left[-\frac{t^4}{4}\right]_0^2$$

$$= e^{-4}$$

$$\approx 0.0183$$

ii)

$$\Pr(0.4 < T < 1.4) = F(1.4) - F(0.4)$$

$$= R(0.4) - R(1.4)$$

$$= \exp[-t^4/4]_0^{0.4} - \exp[-t^4/4]_0^{1.4}$$

$$= \exp[(-1/4)(0.4)^4] - \exp[(-1/4)(1.4)^4]$$

$$= 0.61088$$

iii)

$$\text{MTTF} = \int_0^\infty R(t)dt$$

$$\because\ R(t) = \exp[-\int_0^t \lambda(x)dx]$$

$$= \exp\left[-\int_0^t x^3 dx\right]$$

$$= \exp\left[-\frac{t^4}{4}\right]$$

$$\Rightarrow \text{MTTF} = \int_0^\infty R(t)dt$$

$$= \int_0^\infty \exp(-\frac{1}{4}t^4)dt$$

Let $x = \frac{1}{4}t^4 \ \Rightarrow\ dt = (4x)^{-3/4}dx$

$$\Rightarrow \text{MTTF} = 4^{-3/4}\int_0^\infty e^{-x}\, x^{-3/4}dx$$

$$= 4^{-3/4}\Gamma(1/4)$$

$$\approx 1.2818$$

Q4: [4+4]

(a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

i)

$p_0 = pr(X_0 = 0) = 1$

$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00} P_{00}$

$= 1 \times (1-\alpha) \times (1-\alpha)$

$= (1-\alpha)^2$

ii)

$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$

$= p_0 P_{00} P_{00} + p_0 P_{01} P_{10}$

$= (1-\alpha)^2 + \alpha^2$

$= 1 - 2\alpha + 2\alpha^2$

Q5: [8]

The LP pb. is of form:

$$\begin{array}{ll} \max & 45x_1 + 80x_2 \\ & 5x_1 + 20x_2 \le 400 \\ & 10x_1 + 15x_2 \le 450 \\ & x_1 \ge 0 \ , \ x_2 \ge 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} \max & 45x_1 + 80x_2 \\ & x_1 + 4x_2 \le 80 \\ & 2x_1 + 3x_2 \le 90 \\ & x_1 \ge 0 \ , \ x_2 \ge 0 \end{array}$$

The canonical form:

$$\max \ 45x_1 + 80x_2$$
$$x_1 + 4x_2 + x_3 = 80$$
$$2x_1 + 3x_2 + x_4 = 90$$

where x_3 and x_4 are slack variables.

Let $x_1 = x_2 = 0 \quad \Rightarrow$ NBVs = $\{x_1, x_2\}$ and BVs = $\{x_3, x_4\}$

$$\Rightarrow \quad \begin{aligned} x_3 &= 80 - x_1 - 4x_2 \\ x_4 &= 90 - 2x_1 - 3x_2 \\ z &= 45x_1 + 80x_2 \end{aligned}$$

1st dictionary

Let x_1 be incoming variable (it has a +ve coefficient In the equation for z)

Ratio test

$x_3 : \frac{80}{1} = 80, \quad x_4 : \frac{90}{2} = 45$

$\therefore \ x_4 \rightarrow$ outgoing variable

$\Rightarrow \ x_1 = 45 - 3/2 \ x_2 - 1/2 \ x_4$

$$\Rightarrow \quad \begin{aligned} x_1 &= 45 - 3/2 \ x_2 - 1/2 \ x_4 \\ x_3 &= 35 - 5/2 \ x_2 + 1/2 \ x_4 \\ z &= 2025 + 12.5 \ x_2 - 22.5 \ x_4 \end{aligned}$$

2nd dictionary

Let x_2 be incoming variable (it has a +ve coefficient In the equation for z)

Ratio test

$x_1 : \frac{45}{3/2} = 30, \quad x_3 : \frac{35}{5/2} = 14$

$\therefore \ x_3 \rightarrow$ outgoing variable

$\Rightarrow \ x_2 = 14 - 2/5 \ x_3 + 1/5 \ x_4$

$$\Rightarrow \quad \begin{aligned} x_1 &= 24 + 3/5\ x_3 - 4/5\ x_4 \\ x_2 &= 14 - 2/5\ x_3 + 1/5\ x_4 \\ z &= 2200 - 5\ x_3 - 20\ x_4 \end{aligned}$$

3rd dictionary

Here, we have –ve coefficients for all variables in the z equation, so we should stop.

$\therefore$ The optimal solution is

$x_1 = 24,\ x_2 = 14,\quad x_3 = 0,\ x_4 = 0$ where $z = 2200$

$\therefore$ max $z = \$2200$