King Saud University
Department of Mathematics

First Midterm Exam

2^{nd} **semester 1437 H**

Course Title: Math 425 (Partial Differential Equations)

Date: Monday 15 February 2016; (8-10) am

(. . . . . .) Name . . . . ID . . . .

Question	Grade
Q1	
Q2	
Q3	
Total	

Part I	(a)	(b)	(c)	(d)
Answer	(3)	(2)	(3)	(1)

Question 1

I. Choose the correct answer (write it down on the table above):

(a) A partial differential equation requires

(1) exactly one independent variable
(2) more than one dependent variable
(3) two or more independent variables
(4) equal number of dependent and independent variables.

(b) Using substitution, which of the following equations are solutions to the partial differential equation?

$$\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial^2 u}{\partial y^2}$$

(1) $x^2 + y^2$
(2) $\cos(4x - y)$
(3) $\sin(4x - 4y)$
(4) $e^{-4\pi x}\sin \pi y$.

(c) The solution to the partial differential equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ that satisfies the auxiliary condition $u(0, y) = 5y^2$ is

(1) $u(x, y) = 5(x + y)^2$
(2) $u(x, y) = 5y^2$
(3) $u(x, y) = 5(x - y)^2$
(4) $u(x, y) = 5e^{-2x}y^2$

(d) The characteristic curves of the following PDE are

$$u_x + yu_y = 0$$

(1) $C = ye^{-x}$
(2) $C = \frac{y}{x}$
(3) $C = \frac{y}{x^2}$
(4) $C = xe^{-y}$

II. Classify the following PDEs (linearity, homogeneity and order).

(a) $uu_{xx} + (u_{xy})^2 + u = f(x, y)$

2^{nd} order non-linear inhomogeneous PDE.

(b) $u_{\xi\eta\zeta} + u_{\xi\eta} + u = 0$

3^{rd} order linear homogeneous PDE.

(c) $f(x, y)u_y + g(x, y)u_x + h(x, y)u = h(x, y)$

1^{st} order linear inhomogeneous PDE.

(d) $\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + u = 0$

2^{nd} order quasilinear homogeneous PDE.

Question 2

Find the general solution of the following partial differential equations.

(i) $xp + yq = 2xyz^2$
Quasilinear first order, we use Lagrange method.

The subsidiary equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2xyz^2}$$

1. $\frac{dx}{x} = \frac{dy}{y} \implies \log x = \log y + C \implies x = C_1 y \quad \implies C_1 = \frac{x}{y}$

2. $$\frac{ydx + xdy}{xy + yx} = \frac{dz}{2xyz^2} \implies \frac{ydx + xdy}{\cancel{2xy}} = \frac{dz}{\cancel{2xy}z^2}$$
$\implies d(xy) = \frac{dz}{z^2} \implies xy = -\frac{1}{z} + C_2 \implies \quad C_2 = xy + \frac{1}{z}$

The general solution is

$$F(\frac{x}{y}, xy + \frac{1}{z}) = 0$$

(ii) $5u_x + 4yu_y = 1 + e^{2x}$
1^{st} order linear PDE with variable coefficients.
We let $\xi = x$ and η to be the solution of $\frac{dy}{dx} = \frac{4y}{5}$.

$\frac{dy}{dx} = \frac{4y}{5} \implies 5\frac{dy}{y} = 4dx \implies 5\log y = 4x + C \implies logy = \frac{4}{5}x + C \implies y = C_1 e^{\frac{4}{5}x}$

Then

$$\xi = x \quad \text{and} \quad \eta = e^{-\frac{4}{5}x} y$$

We calculate..

$$u_x = u_\xi - \frac{4}{5} e^{-\frac{4}{5}x} y u_\eta$$
$$u_y = e^{-\frac{4}{5}x} u_\eta$$

Substitute back in the equation, we get

$$\begin{aligned} 5u_\xi - 4e^{-\frac{4}{5}x} y u_\eta + 4e^{-\frac{4}{5}x} y u_\eta &= 1 + e^{2\xi} \\ 5u_\xi &= 1 + e^{2\xi} \\ u_\xi &= \frac{1}{5} + \frac{1}{5} e^{2\xi} \end{aligned}$$

We integrate w.r.t. ξ

$$u(\xi, \eta) = \frac{1}{5}\xi + \frac{1}{10} e^{2\xi} + f(\eta)$$

The general solution is

$$u(x, y) = \frac{1}{5}x + \frac{1}{10} e^{2x} + f(e^{-\frac{4}{5}x} y)$$

(iii) $xyu_x - x^2u_y + yu = 0$
1^{st} order linear PDE with variable coefficients.
We let $\xi = x$ and η to be the solution of $\frac{dy}{dx} = \frac{-x^2}{xy}$.

$\frac{dy}{dx} = \frac{-x^{\cancel{2}}}{\cancel{x}y} \implies ydy = -xdx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \implies C_1 = x^2 + y^2$

Then
$$\xi = x \quad \text{and} \quad \eta = x^2 + y^2$$
We calculate..
$$\begin{aligned} u_x &= u_\xi + 2xu_\eta \\ u_y &= 2yu_\eta \end{aligned}$$
Substitute back in the equation, we get
$$\begin{aligned} xyu_\xi + 2x^2yu_\eta - 2x^2yu_\eta &= 0 \\ xyu_\xi + yu &= 0 \\ u_\xi + \frac{1}{x}u &= 0 \\ u_\xi + \frac{1}{\xi}u &= 0 \end{aligned}$$
Multiply by I.F.$=\exp(\int \frac{1}{\xi}d\xi) = \exp(\log \xi) = \xi$

$$u_\xi + \frac{1}{\xi}u = 0 \implies \xi u_\xi + u = 0 \implies d\big(\xi u\big) = 0 \implies \xi u(\xi,\eta) = f(\eta) \implies u(\xi,\eta) = \frac{1}{\xi}f(\eta)$$

The general solution is $u(x,y) = \frac{1}{x}f(x^2+y^2)$

(iv) $2u_x + 3u_y = \frac{1}{4}\sin x$
1^{st} order linear PDE with constant coefficients.
Let $\xi = 2x + 3y$ and $\eta = 3x - 2y \implies x = \frac{2\xi+3\eta}{13}$ and $y = \frac{3\xi-2\eta}{13}$. We calculate..
$$\begin{aligned} u_x &= 2u_\xi + 3u_\eta \\ u_y &= 3u_\xi - 2u_\eta \end{aligned}$$
Substitute back in the equation, we get
$$\begin{aligned} 4u_\xi + 6u_\eta + 9u_\xi - 6u_\eta &= \frac{1}{4}\sin\big(\frac{2\xi+3\eta}{13}\big) \\ 13u_\xi &= \frac{1}{4}\sin\big(\frac{2\xi+3\eta}{13}\big) \\ u_\xi &= \frac{1}{52}\sin\big(\frac{2\xi+3\eta}{13}\big) \end{aligned}$$
We integrate w.r.t. ξ..
$$u(\xi,\eta) = -(\frac{13}{2})\frac{1}{52}\cos\big(\frac{2\xi+3\eta}{13}\big) + f(\eta)$$
The general solution is $u(x,y) = -\frac{1}{8}\cos x + f(3x-2y)$

Question 3

Determine the integral surface which passes through the given curve:

$$x(y^2+z)P - y(x^2+z)q = (x^2-y^2)z, \quad x(t)=t, \; y(t)=-t \; and \; z=1$$

It is a quasilinear 1^{st} order PDE. We first find the general solution by using Lagrange method then satisfy the condition.

1. Fining the general solution.

   The subsidiary equations

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

   (i)

$$\begin{aligned}\frac{ydx+xdy}{xy^3+\cancel{xyz}-yx^3-\cancel{xyz}} &= \frac{dz}{z(x^2-y^2)}\\ \frac{ydx+xdy}{-xy\cancel{(x^2-y^2)}} &= \frac{dz}{z\cancel{(x^2-y^2)}}\\ \frac{d(xy)}{xy} &= -\frac{dz}{z}\end{aligned}$$

$$\log(xy) = \log z^{-1} + C \implies xy = C_1\tfrac{1}{z} \implies C_1 = xyz.$$

   (ii)

$$\begin{aligned}\frac{xdx+ydy}{\cancel{x^2y^2}+x^2z-\cancel{y^2x^2}-y^2z} &= \frac{dz}{z(x^2-y^2)}\\ \frac{xdx+ydy}{\cancel{z(x^2-y^2)}} &= \frac{dz}{\cancel{z(x^2-y^2)}}\\ xdx+ydy &= dz\end{aligned}$$

$$\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 = z + C \implies x^2+y^2-2z = C_2$$

   The general solution is

$$F(x^2+y^2-2z, xyz) = 0$$

2. Use the condition $x(t)=t, \; y(t)=-t \; and \; z=1$ to find the unique solution.

$$C_1 = xyz = t(-t) = -t^2 \implies t^2 = -C_1$$

   Then

$$\begin{aligned}C_2 &= x^2+y^2-2z\\ &= t^2+t^2-2 = 2t^2-2\\ &= 2(t^2-1) = 2(-C_1-1)\\ &= -2(C_1+1)\end{aligned}$$

$$\implies C_2 = -2(C_1+1).$$

The unique solution is

$$x^2 + y^2 - 2z = -2(xyz + 1)$$