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Question 1:

Define the following:

1) Simply consistent estimator: 2

2) Central limit theorem 2

3) The estimator T* is UMVUE for $\tau(\theta)$ if: 2

1:

2:

4) Efficiency: 2

5) Give the condition for an estimator to be MSE consistent: 2

8

Question 2:

4

6) Does the following density function belong to the exponential family? If yes show?

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\ \Gamma\beta} x^{\alpha-1}(1-x)^{\beta-1} \qquad where \quad 0 \le x \le 1$$

$a(\alpha, \beta) = \frac{\Gamma \alpha+\beta}{\Gamma\alpha \ \Gamma\beta}$ --- 1

$b(x) = 1$ --- 1

$c_1(\alpha, \beta) = \alpha - 1$ - $\frac{1}{2}$ $\quad$ $c_2(\alpha, \beta) = \beta - 1$ - $\frac{1}{2}$

$d_1(x) = \ln(x)$ - $\frac{1}{2}$ $\quad$ $d_2(x) = \ln(1-x)$ — $\frac{1}{2}$

7) Let $X_1, \dots, X_n$ be a random sample from $f(x,\theta) = \theta\, x^{\theta-1}$ where $0 \le x \le 1$ find the estimator using the method of moments.

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$\bar{M}_1 = \bar{\mu}_1$

$\frac{\sum x_i}{n} = E(x)$ — 1

اسكامل — $1\frac{1}{2}$

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$1\frac{1}{2}$

$E(x) = \theta \int_0^1 x^{\theta}\, dx$

$E(x) = \frac{\theta}{\theta+1}$

$\bar{x} = \frac{\theta}{\theta+1}$

$\Rightarrow \hat{\theta} = \frac{\bar{x}}{1-\bar{x}}$

5

σ^{-2}

$-2\sigma^{-3}$

$+ \frac{\cancel{2}}{\cancel{2}\sigma^3} \sum (x_i - \mu)^2$

8) Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ if μ is known find the maximum likelihood estimator for σ^2, σ, $\log \sigma^2$

3 1 1

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} \quad \text{--- } \frac{1}{2}$$

$$\log L = n\log(1) - n\log(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$= -n\log(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2 \quad \text{--- } \frac{1}{2}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2 \quad \text{--- } \frac{1}{2}$$

$$\frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2 = 0 \quad \text{--- } \frac{1}{2}$$

$$\frac{n}{\sigma} = \frac{1}{\sigma^3}\sum(x_i-\mu)^2$$

$$\Longrightarrow \hat{\sigma}^2 = \frac{\sum(x_i-\mu)^2}{n} \quad 1$$

$$\hat{\sigma} = \sqrt{\frac{\sum(x_i-\mu)^2}{n}} \quad \text{--- } 1$$

$$\log \hat{\sigma}^2 = \log \frac{\sum(x_i-\mu)}{n}$$

$$= \log \sum(x_i-\mu) - \log(n) \quad \text{--- } 1$$

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Question 3:

9) State and prove Cramer-Rae inequality:

4 6

$\forall$ T unbaised $\Longrightarrow$ $V(T) \geq \dfrac{[\tau'(\theta)]^2}{E\left(\frac{\partial \ln L}{\partial \theta}\right)}$ -- 1

$E(T) = \int \cdots \int T \cdot L \cdot \prod dx_i = \tau(\theta)$ = $\frac{1}{2}$

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$= \int \cdots \int T \frac{\partial L}{\partial \theta} \prod dx_i = \tau'(\theta)$ — $\frac{1}{2}$

$\frac{\partial}{\partial \theta} L = \frac{1}{L} \frac{\partial L}{\partial \theta} L = \frac{\partial \ln L}{\partial \theta} L$ —— $\frac{1}{2}$

$= \int \cdots \int T \frac{\partial \ln L}{\partial \theta} \cdot L \prod dx_i = E\left(T \cdot \frac{\partial \ln L}{\partial \theta}\right) = \tau'(\theta)$ — $\frac{1}{2}$

$\rho^2\left(T, \frac{\partial \ln L}{\partial \theta}\right) = \mathrm{cov}^2\left(T, \frac{\partial \ln L}{\partial \theta}\right) = \dfrac{\left[E\left(\frac{\partial \ln L}{\partial \theta} \cdot T\right) - E(T)\,E\left(\frac{\partial \ln L}{\partial \theta}\right)\right]^2}{V(T)\,V\left(\frac{\partial \ln L}{\partial \theta}\right)}$ 1

* $E\left(\frac{\partial \ln L}{\partial \theta}\right) = 0$ -- $\frac{1}{2}$

$= \int \cdots \int L \prod dx = 1$ — $\frac{1}{2}$

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$= \int \cdots \int \frac{\partial L}{\partial \theta} \prod dx = 0$ $\frac{1}{2}$

$= \int \cdots \int \frac{\partial \ln L}{\partial \theta} L \prod dx_i = E\left(\frac{\partial \ln L}{\partial \theta}\right) = 0$ — $\frac{1}{2}$

$V\left(\frac{\partial \ln L}{\partial \theta}\right) = E\left(\frac{\partial \ln L}{\partial \theta}\right)^2$ — $\frac{1}{2}$

$\dfrac{(\tau'(\theta))^2}{V(T)\,E\left(\frac{\partial \ln L}{\partial \theta}\right)^2} \leq 1$ — $\frac{1}{2}$

$V(T) \geq \dfrac{(\tau'(\theta))^2}{E\left(\frac{\partial}{\partial \theta} \ln L\right)}$

Cont Q_{10})

$$P(\theta - \epsilon < T_n < \theta + \epsilon) \text{ --- } \frac{1}{2}$$

$$P\left(\frac{\theta - \epsilon}{n+1} < X_{(1)} < \frac{\theta + \epsilon}{n+1}\right) \text{ --- } \frac{1}{2}$$

$$\frac{n}{\theta} \int_{\frac{\theta + \epsilon}{n+1}}^{\frac{\theta+\epsilon}{n+1}} \left[1 - \frac{x}{\theta}\right]^{n-1} dx$$

$$= \frac{n}{\cancel{\theta}} \frac{-\left[1 - \frac{x}{\theta}\right]^{n}}{\frac{n}{\cancel{\theta}}} \Bigg|_{\frac{\theta \mp \epsilon}{n+1}}^{\frac{\theta+\epsilon}{n+1}} \text{ --- } 1$$

$$-\left[1 - \frac{\theta + \epsilon}{(n+1)\theta}\right]^n + \left[1 - \frac{\theta - \epsilon}{(n+1)\theta}\right]^n$$

$$\left(1 - \frac{\theta - \epsilon}{(n+1)\theta}\right)^n + \left(1 - \frac{\theta - \epsilon}{(n+1)\theta}\right)^n \text{ --- } \frac{1}{2}$$

$$= \lim_{n \to \infty} e^{\frac{\theta - \epsilon}{\theta}} - e^{\frac{\theta + \epsilon}{\theta}} \neq 1 \text{ --- } \frac{1}{2}$$

$\Longrightarrow$ not simple con. --- $\frac{1}{2}$

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Question 4:

10) Let $(x;\theta)=\frac{1}{\theta}$ *where* $0\le x\le\theta$. Let T = (n+1)$X_{(1)}$ be an estimator to $\tau(\theta)=\theta$.

Study: 1- unbiased 2- compute MSE 3- consistency

3 $6\frac{1}{2}$ 3.5

$$f_1(x) = n f(x)\,(1-F(x))^{n-1}$$

$$\Longrightarrow f_1(x) = n\frac{1}{\theta}\left(1-\frac{x}{\theta}\right)^{n-1} \qquad 0\le x\le\theta \text{ --- } \frac{1}{2}$$

$$E(X_{(1)}) = \int_0^\theta x\, n\frac{1}{\theta}\left(1-\frac{x}{\theta}\right)^{n-1} dx \text{ --- } \frac{1}{2}$$

let $u=\frac{x}{\theta}$ $\qquad du=\frac{dx}{\theta}$ --- $\frac{1}{2}$

$0\le x\le\theta \Longrightarrow 0\le u\le 1$ --- $\frac{1}{2}$

$$E(X_{(1)}) = n\theta \underbrace{\int_0^1 u^{2-1}(1-u)^{n-1}du}_{\text{Beta}} \text{ --- } \frac{1}{2}$$

$$= n\theta\,\frac{\Gamma_2\,\Gamma_n}{\Gamma_{n+2}} \text{ --- } \frac{1}{2}$$

$$= n\theta\,\frac{(2-1)!\,(n-1)!}{(n+1)!} = \frac{n\theta\,\cancel{(n-1)!}}{n(n+1)\,\cancel{(n-1)!}} \text{ --- } \frac{1}{2}$$

$$E(X_{(1)}) = \frac{\theta}{n+1} \text{ --- } \frac{1}{2}$$

$$E((n+1)X_{(1)}) = n+1\,E(X_{(1)}) = \frac{\theta}{\cancel{(n+1)}}\cancel{(n+1)} = \theta \text{ --- } \frac{1}{2}$$

T is unbiased estimator for $\tau(\theta)$ --- $\frac{1}{2}$

cont Q10) 2) $MSE = V(T) + \underbrace{[E(T) - C(\theta)]^2}$ — $\frac{1}{2}$

$V(T_n) = V((n+1)X_{(1)}) = \overset{\text{zero}}{(n+1)^2 V(X_1)}$ — $\frac{1}{2}$

$= (n+1)^2 \left[E(X_{(1)})^2 - \frac{\theta^2}{(n+1)^2}\right]$ — $\frac{1}{2}$

$E(X_{(1)}^2) = n\int_0^\theta x^2 \frac{1}{\theta}\left(1-\frac{x}{\theta}\right)^{n-1} dx$ - - - $\frac{1}{2}$

$u = \frac{x}{\theta} \quad \frac{dx}{\theta} = du$

$\Rightarrow x = u\theta$ - - $\frac{1}{2}$

$= n \cancel{\theta^2} \int_0^1 u\theta\, u(1-u)^{n-1}\, du\theta$

$= n\theta^2 \int u^2 (1-u)^{n-1}\, du$ — $\frac{1}{2}$

$= n\theta^2 B(3, n)$ — $\frac{1}{2}$

$= n\theta^2 \frac{\Gamma_3 \Gamma_n}{\Gamma_{3+n}}$

$= n\theta^2 \frac{(3-1)!\,(n-1)!}{(n+2)!} = \frac{2\theta^2}{(n+2)(n+1)}$ — $\frac{1}{2}$

$Var(X_{(1)}) = (n+1)^2 \left[\frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2}\right]$ - - $\frac{1}{2}$

$= \cancel{(n+1)^2}\left[\frac{2\theta^2(n+1) - \theta^2(n+2)}{(n+2)\cancel{(n+1)^2}}\right] = \underbrace{\frac{2\theta^2 n + \cancel{2\theta^2} - \theta^2 n - \cancel{2\theta^2}}{n+2}}_{\frac{1}{2}}$

$= \frac{\theta^2 n}{n+2}$ $\frac{1}{2}$

$\lim_{n\to\infty} \frac{\theta^2 n}{n+1} = 1$ $\frac{1}{2}$ $\neq 0$ not MSE consistent $\frac{1}{2}$

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11) Let $f(x;\theta) = \theta e^{-\theta x}$ *where* $x \geq 0$ and $s = \sum x_i$.

Case 1, find: 1- CRLB if $\tau(\theta) = 1/\theta$. (2)

2- UMVUE if the estimator T = $\bar{X}$. (2)

3- Fisher information for s (2)

Case 2, find 1- CRLB if $\tau(\theta) = \theta$. (4)

2- If T = (n-1)/s is efficient. (4)

Case 1

① $CRLB = \frac{[\tau'(\theta)]^2}{I_n \text{ or } E(\frac{\partial \ln L}{\partial \theta})^2 \text{ or} --}$ ---- $\frac{1}{4}$

$f(x,\theta) = \theta e^{-\theta x}$

$L(x,\theta) = \theta^n e^{-\theta \sum x_i}$

$\log L(\theta) = n\log\theta - \theta\sum x_i$ —— $\frac{1}{2}$

$\frac{\partial \log L(\theta)}{\partial\theta} = \frac{n}{\theta} - \sum x_i$ —— $\frac{1}{4}$

$\frac{\partial^2}{\partial^2\theta}\log L(\theta) = \frac{-n}{\theta^2}$

$-\left[E\left(\frac{\partial^2 \ln L}{\partial\theta^2}\right)\right] = \frac{n}{\theta^2}$ —— $\frac{1}{2}$

$\tau'(\theta) = \frac{-1}{\theta^2}$

$\tau'^2(\theta) = \frac{1}{\theta^4}$ $\frac{1}{4}$

$CRLB = \frac{1}{\theta^4} \times \frac{\theta^2}{n} = \frac{1}{\theta^2 n}$ $\frac{1}{4}$

(2)

$E(T) = E(\bar{x}) = E(x) = \frac{1}{\theta}$

$\Longrightarrow$ T unbaised estimator $\frac{1}{2}$

$V(T) = V(\bar{x}) = \frac{\sigma^2}{n} = \frac{1}{n\theta^2}$

$\Longrightarrow$ V(T) = lower boun -- $\frac{1}{2}$

$\Longrightarrow$ T is efficient $\frac{1}{2}$

$\Longrightarrow$ T is UMVUE $\frac{1}{2}$

(3)

$S = \sum X_i \sim Gamma(n,\theta)$

$f(s,\theta) = \frac{\theta^n}{\Gamma n} s^{n-1} e^{-\theta s}$ $\frac{1}{2}$

$\log f(s,\theta) = n\log\theta - \theta s + (n-1)\log s - \log\Gamma n$

$\frac{\partial}{\partial\theta}\log f(s,\theta) = \frac{n}{\theta} - s$ $\frac{1}{4}$

$\frac{\partial^2}{\partial^2\theta}\log f(s,\theta) = \frac{-n}{\theta^2}$ $\frac{1}{4}$

$\underbrace{E\left(-\frac{\partial^2}{\partial\theta^2}\log f(s,\theta)\right) = \frac{n}{\theta^2}}_{\text{الشانون}}$ $\frac{1}{4}$

$\frac{1}{2}$

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Case 2:

① $CRLB = \dfrac{(\tau'(\theta))^2}{-E\left(\frac{\partial}{\partial\theta^2}\log f(x,\theta)\right)}$ → from case 1 $= \dfrac{(1)^2}{\frac{n}{\theta^2}}$ --- $\frac{1}{2}$

$\Longrightarrow CRLB = \dfrac{\theta^2}{n}$ --- $\frac{1}{2}$

4

② $E(T) = (n-1)\, E(\frac{1}{S})$

$E(\frac{1}{S}) = \int_0^\infty \frac{\theta^n}{\Gamma n} s^{n-2} e^{-\theta s} ds$ --- $\frac{1}{2}$

$= \dfrac{\theta^n}{\Gamma n} \dfrac{\Gamma n-1}{\theta^{n-1}} = \dfrac{\theta(n-2)!}{(n-1)!} = \dfrac{\theta}{n-1}$ --- $\frac{1}{2}$

$E(T) = \cancel{(n-1)} \dfrac{\theta}{\cancel{n-1}} = \theta \Longrightarrow T$ is unbaised estimator --- $\frac{1}{2}$

$V(T) = E(T^2) - (E(T))^2$

$= E\left(\dfrac{(n-1)^2}{S^2}\right) - \theta^2 = (n-1)^2 E\left(\dfrac{1}{S^2}\right) - \theta^2$ * --- $\frac{1}{2}$

$E\left(\frac{1}{S^2}\right) = \int_0^\infty \frac{\theta^n}{\Gamma n} s^{n-3} e^{-\theta s} ds$

$= \dfrac{\theta^n}{\theta^{n-2}} \dfrac{\Gamma n-2}{\Gamma n} = \dfrac{\theta^2 (n-3)!}{(n-1)!} = \dfrac{\theta^2}{(n-1)(n-2)}$ $\frac{1}{2}$

* نعوض في

$V(T) = \dfrac{(n-1)^2\theta^2}{\cancel{(n-1)}(n-2)} - \theta^2$

$= \dfrac{(n-1)\theta^2 - (n-2)\theta^2}{(n-2)} = \dfrac{n\theta^2 - \theta^2 - n\theta^2 + 2\theta^2}{(n-2)}$ $\frac{1}{2}$

$\Longrightarrow V(T) = \dfrac{\theta^2}{(n-2)} \neq CRLB$ --- $\frac{1}{2}$

$\Longrightarrow T$ is nat efficient --- $\frac{1}{2}$

درجة (4)

Question 5:

12) Let $f(x\,;\,\theta) = \frac{\ln\theta}{\theta-1}\,\theta^{x}$ *where* $0 < x < 1$

1- Offer a suitable function: $\tau(\theta)$, and an efficient estimator T: (2)

2- Find V(T) and I_n :

(1) (1)

$L(x,\theta) = \left(\frac{\ln\theta}{\theta-1}\right)^{n} \theta^{\sum x_i}$

$\log L = n\ln\left(\frac{\ln\theta}{\theta-1}\right) + \sum x_i \ln\theta$

$= n\ln(\ln\theta) - n\ln(\theta-1) + \sum x_i \log\theta$

$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta\ln\theta} - \frac{n}{\theta-1} + \frac{\sum x_i}{\theta}$

$= \frac{n}{\theta}\left[\frac{1}{\ln\theta} - \frac{\theta}{\theta-1} + \frac{\sum x_i}{n}\right]$

$= \frac{n}{\theta}\left[\frac{\sum x_i}{n} - \left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right)\right]$ $\rightarrow$ $a(n,\theta)\left[T_n - \tau(\theta)\right]$

$T_n = \bar{X}$

$\tau(\theta) = \frac{\theta}{\theta-1} - \frac{1}{\ln\theta}$

$= \frac{\theta\ln\theta - \theta + 1}{(\theta-1)\ln\theta}$

$a(n,\theta) = \frac{n}{\theta}$

$V(T_n) = \frac{\tau'(\theta)}{a(n,\theta)}$

$\Longrightarrow \frac{\theta}{n}\left[\frac{1}{\theta(\ln\theta)^2} - \frac{1}{(\theta-1)^2}\right)$

$\tau'(\theta) = \frac{(\cancel{\theta}-1)-\theta}{(\theta-1)^2} - \frac{-1}{\theta(\ln\theta)^2}$

$\tau'(\theta) = \frac{1}{\theta(\ln\theta)^2} - \frac{1}{(\theta-1)^2}$

$I_n = a(n,\theta)\,\tau'(\theta)$

$\frac{n}{\theta}\left[\frac{1}{\theta(\ln\theta)^2} - \frac{1}{(\theta-1)^2}\right]$

4

Question 6:

13) Prove if T_n is efficient to $\tau(\theta) \Rightarrow V(T_n) = \frac{\tau(\grave{\theta})}{a(n,\theta)}$ & $I_n = a(n,\theta)\tau(\grave{\theta})$

T_n is efficient $\Rightarrow \frac{\partial \ln L}{\partial \theta} = a(n,\theta)(T_n - \tau(\theta))$ --- (1)

$E\left(\frac{\partial}{\partial \theta} \ln L\right)^2 = a(n,\theta)^2 \, E(T_n - \tau(\theta))^2$ $\frac{1}{2}$

$\Longrightarrow I_n = a(n,\theta)^2 V(T_n)$ $\frac{1}{2}$

$\Longrightarrow V(T_n) = \frac{(\grave{\tau}(\theta))^2}{I_n} = \frac{(\grave{\tau}(\theta))^2}{a(n,\theta)^2 V(T_n)}$ (1)

$\Longrightarrow V(T_n) = \frac{\grave{\tau}(\theta)}{a(n,\theta)}$, $I_n = a(n,\theta)\grave{\tau}(\theta)$

$\frac{1}{2}$ $\frac{1}{2}$