King Saud University
Department Of Mathematics.
M-203 [Mid term Examination]
(Differential and Integral Calculus)
(Summer Semester 1431/1432)

Max. Marks: 30 | Time: 2 hrs

Marking Scheme: Q.No:1[3+4+3+5], Q.No:2[4+4+3+4]

Q. No: 1 (a) Discuss the convergence of the sequence $\{n^2 e^{-n}\}$.

(b) Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2}$ converges or diverges.

(c) Use integral test to determine the convergence or divergence of the series $\sum_{n=0}^{\infty} ne^{-n^2}$

(d) Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(x+1)^n$.

Q. No: 2 (a) **Use power series** to approximate the improper integral $\int_0^1 \frac{1-e^{-x}}{x}dx$ **(using first three non-zero terms of the integral) to three decimal places.**

(b) Reverse the order of the integral $\int_0^1 \int_{3y}^{3} e^{x^2} dx dy$ and evaluate.

(c) Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$.

(d) Evalute the integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dydx$ **by changing it to polar co-ordinates.**

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Q#1 (a) Discuss the convergence of the sequence $\{n^2 e^{-n}\}$ [Marks: 3]

Sol: We have $a_n = \frac{n^2}{e^n}$ which we can write as

$$\lim_{n\to\infty} \frac{n^2}{e^n} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \quad (1)$$

Applying L'Hopital rule $\quad \frac{2n}{e^n} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \quad (1)$

Again Applying L'Hopital rule $\quad \frac{2}{e^n} = 0$; Conv. (1)

1(b) Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2}$ converges or diverges. [Marks: 4]

Sol: We use the partial fraction sum:

$$\frac{1}{9n^2+3n-2} = \frac{1}{(3n+2)(3n-1)} = \frac{A}{(3n+2)} + \frac{B}{(3n-1)}$$

$$= \frac{A(3n-1)+B(3n+2)}{(3n+2)(3n-1)} \quad (1)$$

Putting $n = -\frac{2}{3}$, we have $1 = -3A \therefore A = -\frac{1}{3}$

Putting $n = \frac{1}{3}$, we have $1 = 3B \therefore B = \frac{1}{3}$

Hence $\sum_{1}^{n} \frac{1}{3}\left[\frac{1}{3n-1} - \frac{1}{3n+2}\right] = \frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{14}\right)+\dots+\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)\right]$

$$= \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right] \quad (1)$$

$$\lim_{n\to\infty} \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right] = \frac{1}{6}; \text{ Conv.}$$

(1) (1)

(2)

1(c) Use Integral test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} n e^{-n^2}$. [Marks : 3]

Soln. We have $a_n = \frac{n}{e^{n^2}}$ which can be written as

$$f(x) = \frac{x}{e^{x^2}}$$ and we easily see that

$f(x)$ is positive-valued, continuous and decreasing on $[1, \infty)$ (check) (1)

Now, $\lim_{t \to \infty} \int_1^t \frac{x}{e^{x^2}}\,dx$ Put $x^2 = u \Rightarrow 2x\,dx = du$

$$= \lim_{t \to \infty} \frac{1}{2}\int_{1}^{t} \frac{du}{e^u} = \lim_{t \to \infty} -\frac{1}{2}\left[-\frac{1}{e^{x^2}}\right]^t = \frac{1}{2e}.$$ (1) (1) Conv.

1(d) Find the interval of conv. and radius of conv. of the power series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(x+1)^n$. [Marks : 5]

Soln. $\lim_{n \to \infty} \left| \frac{(n+1)^3}{3^{n+1}}(x+1)^{n+1} \times \frac{3^n}{n^3(x+1)^n} \right| = \left|\frac{x+1}{3}\right|$

For abs. conv : $\Leftrightarrow \left|\frac{x+1}{3}\right| < 1 \Leftrightarrow |x+1| < 3$

$\Leftrightarrow -3 < x+1 < 3 \Leftrightarrow -4 < x < 2$ (2)

If $x = -4$, we have $\sum_{n=1}^{\infty} \frac{n^3(-3)^n}{3^n} = \sum_{n=1}^{\infty}(-1)^n n^3$; Div. (1)

If $x = 2$, we have $\sum_{n=1}^{\infty} \frac{n^3(3)^n}{3^n} = \sum_{n=1}^{\infty} n^3$, Div. (1)

$\therefore$ Interval of conv : $(-4, 2)$ (2); Radius of conv : $r = 3$ (1)

(3)

Q# 2(a) Use power series to approximate the integral $\int_0^1 \frac{1-e^{-x}}{x}\,dx$ (using three non-zero terms) to three decimal places. [Marks: 4]

Soln. we have $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$\therefore e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots$ (1)

$\therefore \int_0^1 \frac{1}{x}\left[1 - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right)\right]dx$

$= \int_0^1 \left(1 - \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)dx$ (1)

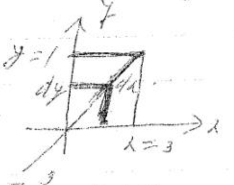
$= \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(3!)} + \cdots\right]_0^1 = 1 - \frac{1}{4} + \frac{1}{18}$ (1)

$= \frac{29}{36} \approx 0.8055$

$\approx \underline{\underline{0.805}}$ (1)

2(b) Reverse the order of the integral $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$ and evaluate the resulting integral. [Mark: 4]

Soln. $\int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 e^{x^2}[y]_0^{x/3}\,dx$ (2)



$= \int_0^3 \frac{1}{3}\left[x e^{x^2}\right]dx$ (1) put $x^2 = u \Rightarrow 2x\,dx = du$

$\frac{1}{6}\left[e^{u}\right] = \frac{1}{6}\left[e^{x^2}\right]_0^3 = \frac{1}{6}\left(e^{9} - 1\right)$ (1)

2(c) Find the surface area of the paraboloid $z = x^2 + y^2$ bounded below by the plane $z = 1$. [Marks: 3]

Soln. $\iint_R \sqrt{1 + 4x^2 + 4y^2}\,dA$ $\quad z = x^2 + y^2 = f(x,y)$ (2) $\therefore f_x = 2x$ and $f_y = 2y$

$= \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2}\, r\,dr\,d\theta$ (2) put $1 + 4r^2 = t^2$, $8r\,dr = dt$, $\frac{1}{8}\int \sqrt{t}\,dt = \frac{1}{8} \times \frac{2}{3} t^{3/2}$

$= \frac{1}{12}\left(5^{3/2} - 1\right)2\pi = \frac{1}{6}\pi\left(5^{3/2} - 1\right)$ (1)

Q # 2(c) Evaluate the Integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ [Marks: 4]

(4)

Soln. $\int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot r\, dr\, d\theta$ (2)

Note: Since $y=0$ is bounded by $y = \sqrt{2x-x^2}$ & is in first quadrant

$y = \sqrt{2x - x^2}$

$\therefore y^2 = 2x - x^2$

$\Rightarrow (x-1)^2 + y^2 = 1$

$\therefore r = 2\cos\theta$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{8}{3}\int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta$$

$$= \frac{8}{3}\int_0^1 (1-t^2)\, dt$$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$ (1)

If $\theta = 0$, $t = 0$ and if $\theta = \pi/2$, $t = 1$

$$= \frac{8}{3}\left[t - \frac{t^3}{3}\right]_0^1$$

$$= \frac{8}{3}\left[1 - \frac{1}{3}\right] = \frac{8}{3} \times \frac{2}{3} = \frac{16}{9}$$ (1)