جامعة
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M-107

Part 1

Linear Algebra

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CHAPTER 1

System of Linear Equations

System of Linear Equations and Matrices

Basic definitions of matrices

1. **Matrix:** A *matrix* is rectangular array of objects, written in rows and columns. These objects can be numbers or functions.

$$\begin{bmatrix} 2 & 1 & 0 \\ -5 & \sqrt{2} & x \end{bmatrix}$$

(This one has 2 Rows and 3 Columns)

2. **Size of a Matrix:** If a matrix A has n rows and m columns, then we say A is " n by m matrix" and we write it as " n x m"

Examples:

(i) $\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$ is 2 x 2 matrix (ii) $\begin{bmatrix} 0 & 1 & 2 \\ 9 & 7 & 4 \\ 3 & 5 & 1 \end{bmatrix}$ is 3x3 matrix

(iii) $\begin{bmatrix} 1 & x & x^2 & e^x \\ x+1 & 2 & 0 & 2x \\ 0 & 0 & 5 & x \end{bmatrix}$ is 3x4 (3 rows x 4 columns) matrix

3. **Square Matrix:** If n = m then the matrix is square matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$ 2x2 is a square matrix

4. **Row Matrix:** $B = \begin{bmatrix} 1 & 2 & 4 & 3 \end{bmatrix}$, 1 x 4 is a row matrix

5. **Column Matrix:** $C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, 4x1 is a column matrix

6. **Zero Matrix:** A zero matrix is a matrix of nay order whose all entries are zero.

$$O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ \text{is a zero matrix.}$$

7. **Diagonal Matrix:** A square matrix with all its non- diagonal entries are zero.

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

8. **Unit Matrix:** A diagonal matrix with all diagonal entries are unity '1'

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

1.1 Linear Equation

$$y = mx \qquad .1$$

is an equation, in which variable y is expressed in terms of x and the constant m, is called Linear Equation.

In Linear Equation exponents of the variable is always ' one'.

Equation 1 is also called equation of line.

Example:1

$$2x+3y=5$$
$$x-y=2$$

are linear equations in two variables and are known as equations of line.

Example:2

$$2x+3y+4z=5$$
$$x-y+2z=2$$

are linear equation in three variables and are known as equations of plane.

1.2Linear Equation in n variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = b \qquad .2$$

where $x_1, x_2, x_3, ..., x_n$ are variables and
$a_1, a_2, a_{13}, ..., a_n$ and b are constants.

1.3 Linear System:

A Linear System of m linear equations and n unknowns is:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2\\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = b_3\\
&\dots\dots\dots\dots\dots\dots\dots\dots \qquad .3\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m
\end{aligned}$$

where $x_1, x_2, x_3, ..., x_n$ are variables or unknowns and a's and b's are constants.

1.4 Solution:

Solution of the linear system (3) is a sequence of n numbers $s_1, s_2, s_3, ..., s_n$, which satisfies system (3) when we substitute $x_1 = s_1, x_2 = s_2, x_3 = s_3, ..., x_n = s_n$.

Example.1. Solve the system of equations

$$x - 3y = -3 \qquad \rightarrow 1$$
$$2x + y = 8 \qquad \rightarrow 2$$

Solution:

$-2E_1 + E_2 \Rightarrow$

$$\begin{aligned}
-2x + 6y &= 6\\
2x + y &= 8\\
\hline
+7y &= 14 \quad \Rightarrow \quad y = 2
\end{aligned}$$

From equation 1

$$x = -3 + 3y$$
$$x = -3 + 6 = 3$$

Solution is $x = 3$ and $y = 2$

Check Substitute the solution in Equations 1 and 2

Equation 1 $\Rightarrow$ $3 - 3(2) = 3 - 6 = -3$

Equation 2 $\Rightarrow$ $2(3) + 2 = 6 + 2 = 8$.

Example.2. Solve the system of equations

$$x - 3y = -7 \qquad \rightarrow 1$$
$$2x - 6y = 7 \qquad \rightarrow 2$$

Solution:

$2E_1 - E_2 \Rightarrow$

$$\begin{array}{r} 2x - 6y = -7 \\ -2x + 6y = -14 \\ \hline 0 + 0 = -21 \end{array}$$

This makes no sense as $0 \neq -21$, hence there is no solution.

NOTE: **Inconsistent** , the system of equations is inconsistent, if the system has no solution.

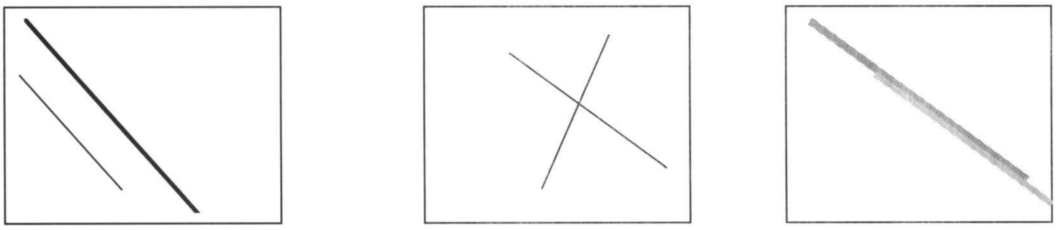
Consistent, the system of equations is consistent if the system has at least one solution.

Example: ***Inconsistent and consistent system of equations***

For the system of linear equations which is represented by straight lines:

$$a_1x \ - \ b_1y \ = c_1 \qquad \rightarrow l_1$$
$$a_2x \ - \ b_2y \ = c_2 \qquad \rightarrow l_2$$

There are three possibilities:



No solution [inconsistent] — one solution [consistent] — infinite many [consistent]

Note:1. A system will have unique solution (only one solution)when number of unknowns is equal to number of equations.

Note:2. A system is over determined , if there are more equations then unknowns and it will be mostly inconsistent.

Note:3. A system is under determined if there are less equations then unknowns and it may turn inconsistent.

1.5 Different Ways of writing System of Linear Equations

Equation Form

System of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_{31} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_{31} = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_{31} = b_3$$

Matrix Form

can be written in the form of matrices product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrix Equation Form

or we may write it in the form AX=b,

where A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, b = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Augmented matrix Form

Augmented matrix is $[A:b] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

Example: 4. Write the matrix and augmented form of the system of linear equations

$$3x - y + 6z = 6$$
$$x + y + z = 2$$
$$2x + y + 4z = 3$$

Solution: Matrix form of the system is

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

Augmented form is $[A:b] = \begin{bmatrix} 3 & -1 & 6 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

1.5 Elementary Row operations:

Elementary row operations are steps for solving the linear system of equations:

I. Interchange two rows
II. Multiply a row with non zero real number
III. Add a multiple of one row to another row

Note: *Elementary row operations produce same results when operated either on a system or on its augmented matrix form.*

1.6 Methods for solving System of Linear equations

1. **Gaussian Elimination Method**
2. **Gauss – Jorden Elimination Method**

1.7 Gaussian Elimination Method

STEP 1. by using elementary row operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & A_{12} & A_{13} & B_1 \\ 0 & 1 & A_{23} & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

STEP 2. Find solution by back – substitutions.

Example:3. Solve the system of linear equations by Gaussion- elimination method

$$x-2y-z=3$$
$$3x-6y-5z=3$$
$$2x-y+z=0$$

Solution: Augmented matrix is

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

STEP 1. Creating 0 in the first below first entry by performing row operations $-3R_1+R_2 \Rightarrow R_2$, $-2R_1+R_3 \Rightarrow R_3$

$$\approx \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & 3 & 3 & -6 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & -2 & -6 \end{bmatrix} R_2 \Leftrightarrow R_3$$

Creating 1 in second entry of the second row and in third entry of the third row by performing row operations $\frac{1}{3}R_2 \Rightarrow R_2, -\frac{1}{2}R_3 \Rightarrow R_3$

$$\approx \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Equivalent system of equations form is:

$$x-2y-z=3$$
$$y+z=-2$$
$$z=3$$

STEP 2. **Back Substitution**

$$z=3$$
$$y=-z-2=-3-2=-5$$
$$x=2y+z+3=-10+3+3=-4$$

Solution is

$$x=-4,\ y=-5,\ z=3$$

Example 4. Suppose that points (-2,-1), (-1,2), (1,2) lie on parabola

$$y = a + bx + cx^2,$$

(i) Determine a linear system of equations in three unknown a, b and c,
(ii) Find the equation of parabola by solving the system of linear equation.

Solution:

(i) The system of linear equations can be obtained by substituting these points in the equation of parabola as these lie on the parabola.

$$\begin{array}{ll} \text{Through point } (-2,-1) & a - 2b + 4c = -1 \\ \text{through point } (-1,2) & a - b + c = 2 \\ \text{through point } (1,2) & a + b + c = 2 \end{array}$$

The system of linear equations is

$$\begin{aligned} a - 2b + 4c &= -1 \\ a - b + c &= 2 \\ a + b + c &= 2 \end{aligned}$$

(ii)

STEP: I.

Matrix form of the system is:

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Augmented matrix form is:

$$\begin{bmatrix} 1 & -2 & 4 & -1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Creating 0 in the first below first entry by performing row operations
$-R_1 + \mathbf{R_2}$ **and** $-R_1 + \mathbf{R_3}$

$$\approx \begin{bmatrix} 1 & -2 & 4 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 3 & -3 & 3 \end{bmatrix}$$

Creating 0 in second entry of the third row by performing row operations
$-3R_2+\mathbf{R_3}$

$$\approx \begin{bmatrix} 1 & -2 & 4 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

STEP: II.

We are using Gauss Elimination method , so we write the equation of the matrix

$$\begin{aligned} a - 2b + 4c &= -1 \\ -b - 3c &= 3 \\ 6c &= -6 \end{aligned}$$

Solving by backward substitution

$$6c = -6 \quad \Rightarrow c = -1$$

$$-b = 3c + 3 = -3 + 3 = 0 \qquad \Rightarrow b = 0$$

$$a = 2b - 4c - 1 = 0 + 4 - 1 = 3 \quad \Rightarrow a = 3$$

Solution of the system *is* $a = 3,\ b = 0$ and $c = -1$

Equation of parabola is $y = 3 - x^2$

1.8 Gauss – Jorden Elimination Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

Example.4. Solve the system of linear equations by Gauss - Jorden elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Solution: Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \qquad \mathbf{R_1+R_2,\ \ -3R_1+R_3}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \qquad \mathbf{-R_2,\ \ 10R_2+R_3}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad -R_3/52$$

$$\approx \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad -2R_3+R_1,\ \ 5R_3+R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad -R_2+R_1$$

Equivalent system of equations form is:

$x_1 = 3$
$x_2 = 1$
$x_3 = 2$ is the solution of the system.

1.9 Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. Below the leading entry all values must be zero,
iv. A row containing all zero values must be in the bottom.

Examples:

$$(i) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad (ii) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (iii) \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1.10 Reduced Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. All entries in the column containing leading entry must be zero,
iv. A row containing all zero values must be in the bottom.

Examples

$$(i) \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad (ii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, (iii) \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example:5. Use Gauss – Jorden method to solve the system of linear system

$$\begin{aligned} x - \ \ y + 2z - \ w &= -1 \\ 2x + \ \ y - 2z - 2w &= -2 \\ -x + 2y - 4z + \ w &= \ 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

Solution: Gauss-Jorden method is same as to reduce the augmented matrix to reduced row echelon from.

Augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

There is a leading entry '1' in the first row, making all other entries in the first column zero

$$\approx \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \quad (-2R_1+R_2)/3 \ , \ R_1+R_3 \ , -3R_1+R_4$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2+R_1 \ , \ -R_2+R_3 \ , \ -3R_2+R_4$$

is reduced row echelon form

Equivalent matrix form is

$$x - w = -1$$

$$y - 2z = 0$$

there are four variables x, y, w and z in the example, variables appearing as leading entries are called LEADING VARIABLES, and other variables are FREE VARIABLE

x and y are leading variables and w and z are free variables.

Let $z = s$ and $w = t$, where s and t are real numbers ,

$$\begin{aligned} x &= -1 + w = -1 + t \\ y &= 2z \qquad = 2s \\ z &= s \\ w &= t, \quad t \text{ and } s \in R \end{aligned}$$

There are infinite many solutions of the given system.

1.11 SYSTEM *WITH NO SOLUTION*

Example: 6 . Solve the system of linear equations

$$
\begin{aligned}
x - 2y + z - 4u &= 1 \\
x + 3y + 7z + 2u &= 2 \\
x - 12y - 11z - 16u &= 5
\end{aligned}
$$

Solution:

Augmented matrix is:

$$
\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix}
$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$
\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \qquad R_2 - R_1,\ R_3 - R_1
$$

$$
\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \qquad -R_3 + 2R_2
$$

Last equation is

$$0x + 0y + 0z + 0u = 6$$

but $\quad 0 \neq 6$

hence there is no solution for the given system of linear equations.

EXERCISE 1.1

SYSYTEM OF LINEAR EQUATIONS

Solve the system of linear equations
(a) Gaussian Elimination (Row Echelon form)
(b) Gauss – Jordon method (Reduced Row Echelon form)

1.
$$\begin{aligned} x+y+z &= 7 \\ -x+y+z &= 5 \\ x-y+z &= 5 \end{aligned}$$
Ans: $x=1, y=1, z=5$

2.
$$\begin{aligned} x+2y-z &= 2 \\ 2x+3y+2z &= 5 \\ 3x+2y+3z &= 10 \end{aligned}$$
Ans: $x=4, y=-1, z=0$

3.
$$\begin{aligned} 3x+y-z &= -4 \\ x+y-2z &= -4 \\ -x+2y-z &= 1 \end{aligned}$$
Ans: $x=-1, y=1, z=2$

4.
$$\begin{aligned} x+2y+3z &= 6 \\ 2x-3y+2z &= 14 \\ 3x+y-z &= -2 \end{aligned}$$
Ans: $x=1, y=-2, z=3$

5.
$$\begin{aligned} x-y+z &= 4 \\ 4x+2y+2z &= 8 \\ 3x+2y+2z &= 2 \end{aligned}$$
Ans: $x=6, y=-3, z=-5$

6.
$$\begin{aligned} x+2y+3z &= 17 \\ 3x+2y+z &= 11 \\ x-5y+3z &= -5 \end{aligned}$$
Ans: $x=1, y=2, z=4$

7.

$$x+8y+2z=7$$
$$2x+4y-4z=3$$
$$2x+y+z=2 \qquad \text{Ans: } x=1/2, y=3/4, z=1/4$$

8.

$$x+y-2z=1$$
$$2x-y+z=2$$
$$x-2y-4z=-4 \qquad \text{Ans: } x=26/21, y=25/21, z=5/7$$

Solve the system of linear equations by Gauss – Jordan method

9.
$$4x_1 - x_3 + 2x_4 = -3$$
$$x_2 + x_3 + 3x_4 = 1$$
$$x_1 - 4x_2 + 3x_3 + x_4 = 0$$

10.
$$x_1 + 2x_2 + 4x_3 + x_4 = 3$$
$$2x_1 + 3x_3 = 5$$
$$x_1 + 3x_2 + 2x_4 = 2$$

11.
$$x_1 - 2x_2 + x_3 - 3x_4 = 2$$
$$x_1 + 3x_2 + 7x_3 + 2x_4 = 3$$
$$x_1 - 12x_2 - 11x_3 - 13x_4 = 7$$

12.
$$x_1 - x_2 + 2x_3 + x_4 = -1$$
$$2x_1 + x_2 + x_3 - x_4 = 4$$
$$x_1 + 2x_2 - x_3 - 2x_4 = 5$$
$$x_1 + x_3 = 1$$

13.

$$x+2y+3z+4w=5$$
$$x+3y+5z+7w=11$$
$$x - z - 2w = -6$$

Ans: No solution

14.

$$x+y+2z-5w=3$$
$$2x+y-z+3w=-11$$
$$2x+5y-z-9w=-3$$
$$x-3y+2z+7w=-5$$

Ans: $x=-5-2t, y=2+3t, z=3+2t, w=t$

15.

$$2x_1 + x_2 + 6x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 2x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10$$

Ans: $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$

16.

$$2x - 3y - z = 5$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$3x + 3y + 2z = 1$$

Ans: $x = 2, y = 1, z = -4$

17.

$$3x - y + 7z = 0$$
$$2x - y + 4z = \frac{1}{2}$$
$$x - y + z = 1$$
$$6x - 4y + 10z = 3$$

Ans: $x = -\frac{1}{2} - 3t, y = -\frac{3}{2} - 2t, z = t,$ where t is arbitrary.

18.

$$x_1 + x_2 - x_3 + 2x_4 = 10$$
$$-5x_1 + 3x_2 - 15x_3 - 6x_4 = 9$$
$$3x_1 - x_2 + 7x_3 + 4x_4 = -1$$

Ans: System is inconsisitent.

1.12 *Conditions on Solutions*

Example:7. For which values of 'a' will be following system

$$\begin{aligned} x+2y-3z &= 4 \\ 3x-y+5z &= 2 \\ 4x+y+(a^2-14)z &= a+2 \end{aligned}$$

(i) infinitely many solutions?
(ii) No solution?
(iii) Exactly one solution?

Solution:

Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$$

Reducing it to row echelon form

$$\approx \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & -14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \quad R_2\text{-}3R_1,\ R_3\text{-}4R_1$$

$$\approx \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} \quad -\tfrac{1}{7}R_2,\ R_3\text{-}R_2$$

writing in the equation form,

$$\begin{aligned} x+2y-3z &= 4 && \to 1 \\ y-2z &= \tfrac{10}{7} && \to 2 \\ (a^2-16)z &= a-4 && \to 3 \end{aligned}$$

or equation 3 can be written as

$$(a+4)(a-4)z = a-4$$

CASE I .

$$a = 4 \quad \Rightarrow 0z = 0$$

$$x + 2y - 3z = 4$$
$$y - 2z = \tfrac{10}{7}$$

as number of equations are less than number of unknowns, hence the system has infinite many solutions,

CASE II

$a = -4 \quad \Rightarrow \quad 0z = -8$, but $0 \neq -8$, hence, there is no solution.

CASE III

$$a \neq 4, a \neq -4, \quad \text{let} \quad a = 1$$

Equatins.3. $\Rightarrow (1-4)(1+4)z = 1-4$

$$-15z = -3$$
$$z = \tfrac{1}{5}$$

$$y = \tfrac{10}{7} + \tfrac{2}{5} = \tfrac{64}{35}$$
$$x = 4 + \tfrac{3}{5} - 2(\tfrac{64}{35}) = \tfrac{47}{35}$$

the system will have unique solution when $a \neq 4$ and $a \neq -4$
and for $a=1$ the solution is

$x = \tfrac{47}{35}, y = \tfrac{64}{35}$ and $z = \tfrac{1}{5}$.

NOTE: (i) $a=-4$, no solution,
(ii) $a=4$, infinite many solutions and
(iii) $a \neq 4$, $a \neq -4$, exactly one solution .

Example:8. What conditions must a, and b satisfy in order for the system of equations

$$\begin{aligned} x-2y+3z &= 4 \\ 2x-3y+az &= 5 \\ 3x-4y+5z &= b \end{aligned}$$

to have (i)infinitely many solutions? (ii) No solution? (iii) Exactly one solution?

Solution: The augmented matrix is

$$[A:B]=\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix}$$

reducing it to reduced row echelon form

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{bmatrix} \quad R_2-2R_1,\ R_3-3R_1$$

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{bmatrix} \quad R_3-2R_2$$

(i) Infinitely many solutions?

If $a=4$ and $b=6$ then

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) No solution?

If $a=4$ and $b\neq 6$ then

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & b-6 \end{bmatrix}$$

(iii) Exactly one solution?

If $a\neq 4$ and $b\in R$ then

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \dfrac{b-6}{-2a+8} \end{bmatrix}$$

Example:9. What conditions must a, b, and c satisfy in order for the system of equations

$$\begin{aligned} x + y + 2z &= a \\ x \quad\quad + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

to be consistent.

Solution: The augmented matrix is

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$ reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \quad R_2\text{-}R_1,\ R_3\text{-}2R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{bmatrix} \quad R_3\text{-}R_1$$

The system will be consistent if only if $c - a - b = 0$

Or $c = a + b$

Thus the required condition for system to be consistent is

$c = a + b.$

EXERCISE 1.2

Conditions on solutions

Find the relationship between a, b and c for which the system of equations will be consistent:

1.
$$\begin{aligned} 2x - y - z &= a \\ x + 2y + z &= b \\ 5x + 4y - z &= c \end{aligned}$$

2.
$$\begin{aligned} x_1 - x_2 + x_3 &= a \\ 2x_1 - x_2 + 3x_3 &= b \\ x_1 \quad\;\; + 2x_3 &= c \end{aligned}$$

3.
$$\begin{aligned} x_1 - 2x_2 + x_3 &= a \\ 2x_1 + x_2 + x_3 &= b \\ 5x_2 - x_3 &= c \end{aligned}$$

4.
$$\begin{aligned} x_1 - x_2 + 3x_3 &= a \\ 3x_1 - 3x_2 + 9x_3 &= b \\ -2x_1 + 2x_2 - 6x_3 &= c \end{aligned}$$
Ans: $2b = c + 4a$

5.
$$\begin{aligned} x + 2y - 2z - 2t &= a \\ -y + 3z + 2t &= b \\ -x + y + 4z + 3t &= c \\ 4y - z - t &= 2 \end{aligned}$$
Solution: $a - b + c = 2$

$ans.\ x = -\frac{3}{2}t, y = \frac{1}{2}t, z = t$

For what values of λ does the following system of linear equations have (a) Unique solution, (b) infinite many solutions, and (c) no solution.

6.
$$\begin{aligned} 3x \quad\quad + \lambda z &= 2 \\ 3x + 3y + 4z &= 4 \\ y + 2z &= \lambda \end{aligned}$$

Ans. , (a) $\lambda \neq -2$,$(b)\lambda = -2$ and $\lambda = 2/3$, (c) $\lambda = -2$.

7.
$$\begin{aligned} x + y + z &= 3 \\ 2x - y - z &= 1 \\ 3x + \lambda y + \lambda^2 z &= 7 \end{aligned}$$

Ans. , (a)$R - \{0,1\}$,$(b)\lambda = 9/5$, (c) $\lambda = 0\ or \lambda = 1$

8. $$\begin{aligned} 2x + 3y + z &= -1 \\ x + 2y + z &= 0 \\ 3x + y + (\lambda^2-13)z &= \lambda-3 \end{aligned}$$

Ans. ,(a) $\lambda \neq \pm\sqrt{11}$,(b)$\lambda = \pm\sqrt{11}$ and $\lambda = -2$,(c)$\lambda = \pm\sqrt{11}$

9. $$\begin{aligned} x + y + 2z &= \lambda \\ x - 3z &= \lambda^2 \\ 2x + y - z &= 0 \end{aligned}$$

Ans. (a) No unique solution,(b) $\lambda = 0$ or $\lambda = -1$, (c) $\lambda \neq -1, \lambda \neq 0$.

10. $$\begin{aligned} 2x + y - z &= 1 \\ x + y + z &= 2 \\ 3x + 2y &= \lambda \end{aligned}$$

11. $$\begin{aligned} x + 2y - 4z &= 3 \\ 3x - y + 13z &= 3 \\ 4x + y + \lambda^2 z &= \lambda+3 \end{aligned}$$

12. $$\begin{aligned} x + y - z &= 2 \\ x + y + z &= 3 \\ x + y + (\lambda^2-5)z &= \lambda \end{aligned}$$

Ans. , (a) $\lambda = 2$,(b)$\lambda \neq \pm 2$, (c) $\lambda = -2$.

For what values of λ and μ does the following system of linear equations have (a)Unique solution, (b) infinite many solutions, and (c) no solution.

13.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

$Ans:$ $(a)\ \lambda \neq 3\ \mu \in R,\ (b)\lambda = 3, \mu = 10\ (c)\ \lambda = 3, \mu \neq 10$

14.

$$\begin{aligned} x - 2y + 3z &= 4 \\ 2x - 3y + \lambda z &= 5 \\ 3x - 4y + 5z &= \mu \end{aligned}$$

15. For what values of λ does the following system of linear equations have(a) Unique solution, (b) infinite many solutions, and (c) no solution.

$$\begin{aligned} x+y+z&=2 \\ 2x+y-z&=4 \\ 3x+2y\quad&=\lambda \end{aligned}$$

Ans: $\lambda = 6$, *system will have many solutions*

16. For what values of λ does the following system has non-trivial solution

$$\begin{aligned} 2x+y-z&=1 \\ x+y+z&=2 \\ 3x+2y\quad&=\lambda \end{aligned}$$

17.**Find a relation between a and c so that the system**

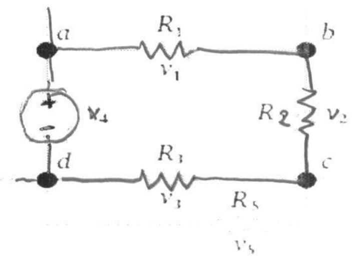
$$\begin{aligned} ax+3z&=2 \\ 3y-2z&=1 \\ x+cz&=2 \end{aligned}$$ has a unique solution

1.13 Applications

1.13.1 Kirchhoff's Voltage law (Kirchhoff's Loop rule)

In a closed loop within an electric circuit, the sum of the voltage drops in any one direction equals the sum of the voltage source in the same direction.

$$\Sigma I\, R = V$$



- represents battery
- denotes resistor
- The currents flows from -ve to +ve

$$V_1 + V_2 + V_3 = V_4.$$

Ohm's Law

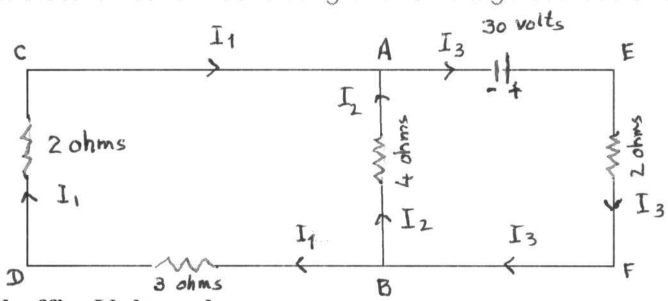
V=I R , where I is Current in amperes, R is Resistance in ohms, and V is volts drop across the resister.

Kirchhoff's Current law

The current in any junction equals the current out of junction.

Example 1.

Assume an electric network consisting of one voltage sources and three resistors



1. **Kirchhoff's Voltage law**

Loop BDCAB $\quad 3I_1 + 2I_1 - 4I_2 = 0$

Loop BAEFB $\quad 4I_2 + 2I_3 = 30$

Loop DCAEBFD $\quad 2I_1 + 2I_3 + 3I_1 = 30$

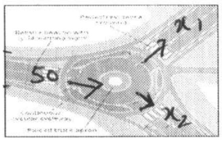
2. **Kirchhoff's Current law**

At A $\quad I_1 + I_2 - I_3 = 0$

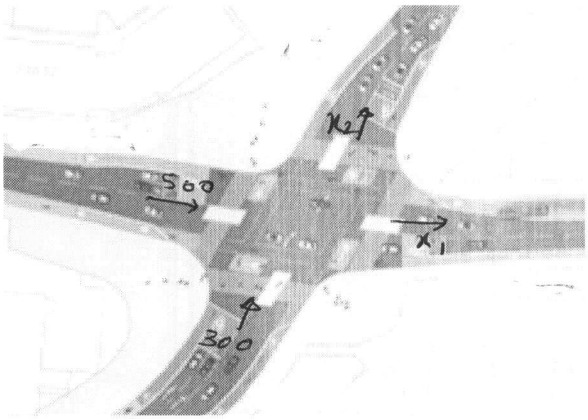
At B $\quad I_3 - I_1 - I_2 = 0$

1.13.2 Road Network Analysis

The total flow inside a junction is equal to the total flow out side the junction.
Example:



$$50 = x_1 + x_2$$



$$500 + 300 = x_1 + x_2$$

1.13.3 Polynomial Curve Fitting

Example: Write linear equation for the polynomial

$P(x) = ax^2 + bx + c$ passing through points $(1,5), (2,4)$ **and** $(3,10)$.

$$P(1) = a(1)^2 + b(1) + c = a + b + c = 5$$

$$P(2) = a(2)^2 + b(2) + c = 4a + 2b + c = 4$$

$$P(1) = a(3)^2 + b(3) + c = 9a + 3b + c = 10$$

Note: To find polynomial, we solve the linear equations with varibles a, b and c.

EXAMPLE The network in Fig. 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

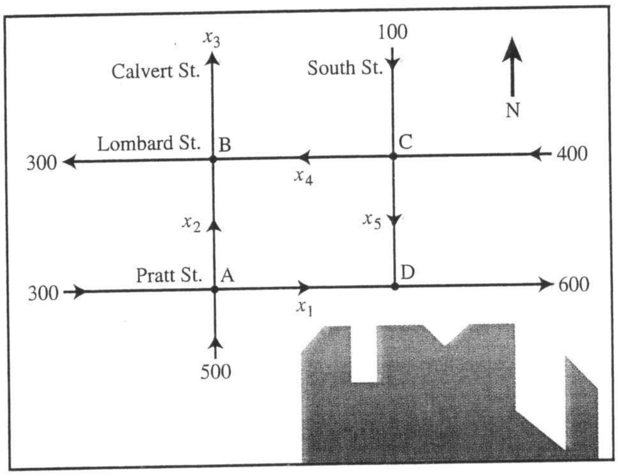


FIGURE 2 Baltimore streets.

SOLUTION Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in Fig. 2. At each intersection, set the flow in equal to the flow out.

Intersection	Flow in		Flow out
A	$300 + 500$	=	$x_1 + x_2$
B	$x_2 + x_4$	=	$300 + x_3$
C	$100 + 400$	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600

Also, the total flow into the network $(500 + 300 + 100 + 400)$ equals the total flow out of the network $(300 + x_3 + 600)$, which simplifies to $x_3 = 400$. Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:

$$
\begin{array}{rcrcrcrcrcl}
x_1 & + & x_2 & & & & & & & = & 800 \\
 & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\
 & & & & & & x_4 & + & x_5 & = & 500 \\
x_1 & & & & & & & + & x_5 & = & 600 \\
 & & & & x_3 & & & & & = & 400
\end{array}
$$

Row reduction of the associated augmented matrix leads to

$$
\begin{array}{rcrcrcrcrcl}
x_1 & & & & & & & + & x_5 & = & 600 \\
 & & x_2 & & & & & - & x_5 & = & 200 \\
 & & & & x_3 & & & & & = & 400 \\
 & & & & & & x_4 & + & x_5 & = & 500
\end{array}
$$

The general flow pattern for the network is described by

$$
\begin{cases}
x_1 = 600 - x_5 \\
x_2 = 200 + x_5 \\
x_3 = 400 \\
x_4 = 500 - x_5 \\
x_5 \text{ is free}
\end{cases}
$$

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one-way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ because x_4 cannot be negative. Other constraints on the variables are considered in Practice Problem 2. ■

> **KIRCHHOFF'S VOLTAGE LAW**
>
> The algebraic sum of the RI voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.

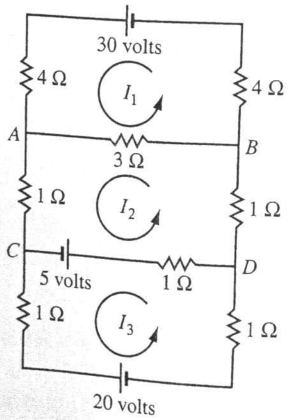


FIGURE 1

EXAMPLE 2 Determine the loop currents in the network in Fig. 1.

SOLUTION For loop 1, the current I_1 flows through three resistors, and the sum of the RI voltage drops is

$$4I_1 + 4I_1 + 3I_1 = (4 + 4 + 3)I_1 = 11I_1$$

Current from loop 2 also flows in part of loop 1, through the short *branch* between A and B. The associated RI drop there is $3I_2$ volts. However, the current direction for the branch AB in loop 1 is opposite to that chosen for the flow in loop 2, so the algebraic sum of all RI drops for loop 1 is $11I_1 - 3I_2$. Since the voltage in loop 1 is $+30$ volts, Kirchhoff's voltage law implies that

$$11I_1 - 3I_2 = 30$$

The equation for loop 2 is

$$-3I_1 + 6I_2 - I_3 = 5$$

The term $-3I_1$ comes from the flow of the loop-1 current through the branch AB (with a negative voltage drop because the current flow there is opposite to the flow in loop 2). The term $6I_2$ is the sum of all resistances in loop 2, multiplied by the loop current. The term $-I_3 = -1 \cdot I_3$ comes from the loop-3 current flowing through the 1-ohm resistor in branch CD, in the direction opposite to the flow in loop 2. The loop-3 equation is

$$-I_2 + 3I_3 = -25$$

Note that the 5-volt battery in branch CD is counted as part of both loop 2 and loop 3, but it is -5 volts for loop 3 because of the direction chosen for the current in loop 3. The 20-volt battery is negative for the same reason.

The loop currents are found by solving the system

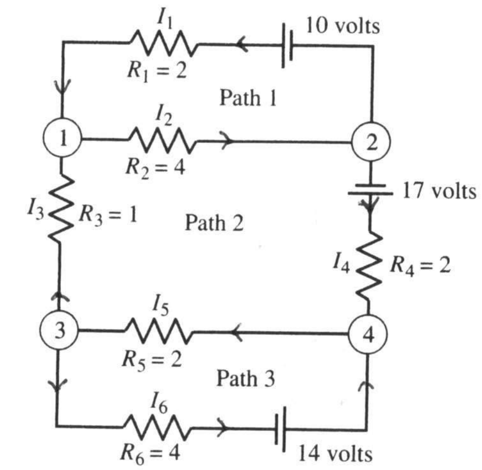
$$\begin{aligned} 11I_1 - 3I_2 \phantom{{}- I_3} &= 30 \\ -3I_1 + 6I_2 - I_3 &= 5 \\ - I_2 + 3I_3 &= -25 \end{aligned} \tag{3}$$

Row operations on the augmented matrix lead to the solution: $I_1 = 3$ amps, $I_2 = 1$ amp, and $I_3 = -8$ amps. The negative value of I_3 indicates that the actual current in loop 3 flows in the direction opposite to that shown in Fig. 1. ■

Example: 3

Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for th

Figure 1.14



Solution Applying Kirchhoff's first law to the four junctioı

$$
\begin{aligned}
I_1 + I_3 &= I_2 && \text{Junction 1}\\
I_1 + I_4 &= I_2 && \text{Junction 2}\\
I_3 + I_6 &= I_5 && \text{Junction 3}\\
I_4 + I_6 &= I_5 && \text{Junction 4}
\end{aligned}
$$

and applying the second law to the three paths prc

$$
\begin{aligned}
2I_1 + 4I_2 &= 10 && \text{Path 1}\\
4I_2 + I_3 + 2I_4 + 2I_5 &= 17 && \text{Path 2}\\
2I_5 + 4I_6 &= 14. && \text{Path 3}
\end{aligned}
$$

Thus we have the following system of seven linear equ

$$
\begin{array}{rcl}
I_1 - I_2 + I_3 & = & 0\\
I_1 - I_2 + I_4 & = & 0\\
I_3 - I_5 + I_6 & = & 0\\
I_4 - I_5 + I_6 & = & 0\\
2I_1 + 4I_2 & = & 10\\
4I_2 + I_3 + 2I_4 + 2I_5 & = & 17\\
2I_5 + 4I_6 & = & 14
\end{array}
$$

Using Gauss-Jordan elimination, a calculator, or

$I_1 = 1, \quad I_2 = 2 \quad I_3 = 1, \quad I_4 = 1,$

which means that $I_1 = 1$ amp, $I_2 = 2$ amps, $I_3 = 1$ amp,

1.13.4

Temperature Distribution

A simple model for estimating the temperature distribution on a square plate gives rise to a linear system of equations. To construct the appropriate linear system, we use the following information: The square plate is perfectly insulated on its top and bottom so that the only heat flow is through the plate itself. The four edges are held at various temperatures. To estimate the temperature at an interior point on the plate, we use the rule that it is the average of the temperatures at its four compass-point neighbors, to the west, north, east, and south.

EXAMPLE 8

Estimate the temperatures T_i, $i = 1, 2, 3, 4$, at the four equispaced interior points on the plate shown in Figure 2.2.

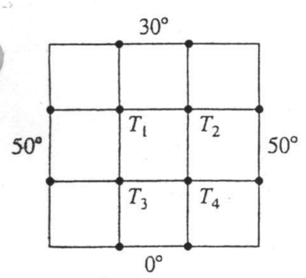


FIGURE 2.2

Solution

We now construct the linear system to estimate the temperatures. The points at which we need the temperatures of the plate for this model are indicated in Figure 2.2 by dots. Using our averaging rule, we obtain the equations

$$\begin{aligned}
T_1 &= \frac{50+30+T_2+T_3}{4} &\text{or}\quad& 4T_1 - T_2 - T_3 = 80\\
T_2 &= \frac{T_1+30+50+T_4}{4} &\text{or}\quad& -T_1 + 4T_2 - T_4 = 80\\
T_3 &= \frac{50+T_1+T_4+0}{4} &\text{or}\quad& -T_1 + 4T_3 - T_4 = 50\\
T_4 &= \frac{T_3+T_2+50+0}{4} &\text{or}\quad& -T_2 - T_3 + 4T_4 = 50.
\end{aligned}$$

The augmented matrix for this linear system is (verify)

$$\left[A \mid \mathbf{b}\right] = \left[\begin{array}{rrrr|r} 4 & -1 & -1 & 0 & 80\\ -1 & 4 & 0 & -1 & 80\\ -1 & 0 & 4 & -1 & 50\\ 0 & -1 & -1 & 4 & 50 \end{array}\right].$$

Using Gaussian elimination or Gauss–Jordan reduction, we obtain the unique solution (verify)

$$T_1 = 36.25°,\quad T_2 = 36.25°,\quad T_3 = 28.75°,\quad \text{and}\quad T_4 = 28.75°. \quad \blacksquare$$

Applications

1. Suppose that points (1,1), (2,3), (3,-1) lie on parabola

$$y = a + bx + cx^2$$

,

(i) Determine a linear system of equations in three unknown a, b and c,

(ii) Find the equation of parabola by solving the system of linear equation.

2. Find the polynomial function

$$P(x) = ax^2 + bx + c$$

$$\text{such that } P(0) = 1, P(1) = 4, P(2) = 11$$

Ans: $a = 2,\ b = 1, c = 1$

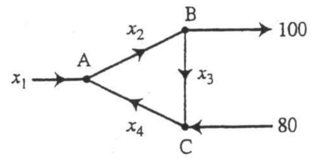
3. By applying Kirchoff's law to circuit we obtain the following system of linear equations

$$\begin{aligned} i_1 + 2i_2 - 7i_3 &= -4 \\ 2i_1 + i_2 + i_3 &= 13 \\ 3i_1 + 9i_2 - 36i_3 &= -33 \end{aligned}$$

Find the current in the circuit.

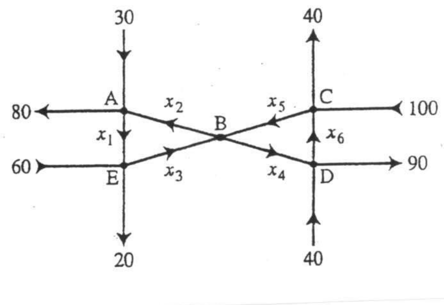
Ans: $i_1 = 10 - 3t, i_2 = -7 + 5t, i_3 = t$

4. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for x_4?

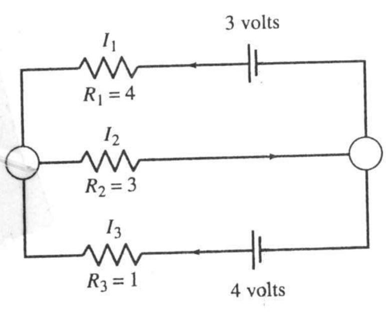


5 a. Find the general flow pattern of the network shown in the figure.

b. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by x_2, x_3, x_4, and x_5.



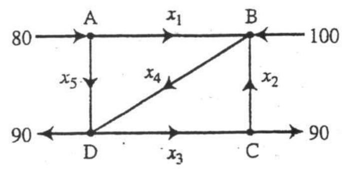
C 23. Determine the currents I_1, I_2, and I_3 for the electrical network



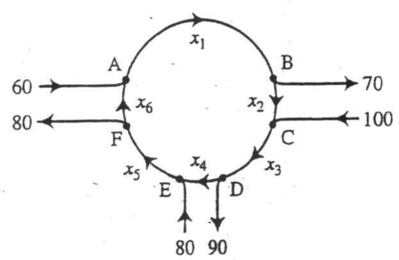
6 a. Find the general traffic pattern of the freeway network shown in the figure. (Flow rates are in cars/minute.)

b. Describe the general traffic pattern when the road whose flow is x_5 is closed.

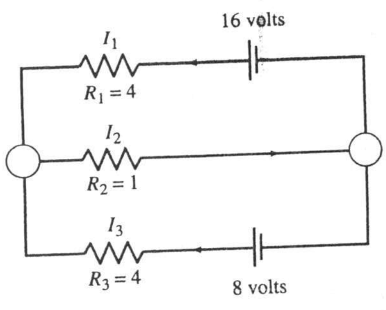
c. When $x_5 = 0$, what is the minimum value of x_4?



7. Intersections in England are often constructed as one-way "roundabouts," such as the one shown in the figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for x_6.



C 24. Determine the currents I_1, I_2, and I_3 for the electrical network



1.14 HOMOGENEOUS SYSTEM

Let AX= b be asystem of linear equations. This system is called **homogeneous system of linear equations** if and only if b = 0.
A system of equations of the form

AX = 0.,

That is with all constants b's taken as zero.

Example:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots. + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots. + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots. + a_{3n}x_n = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots. + a_{mn}x_n = 0$$

is homogeneous system of linear equations.

1. The homogeneous system AX=0 always has at least one solution, $x_1 = x_2 = x_3 = \ldots. = x_n = 0$ called a **trivial solution.**

2. The homogeneous system has infinitely many **non- trivial solutions** in addition to the trivial solutions.

3. The homogeneous system will have a non- trivial solution if and only if A is a singular matrix $\Rightarrow$ determinant of A is zero.

Example:10. Solve the homogeneous system of linear equations

$$\begin{aligned} 2x+2y+4z&=0\\ w \quad\quad - y-3z&=0\\ 2w+3x+\ y+\ z&=0\\ -2w+\ x+3y-2z&=0 \end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0\\ 1 & 0 & -1 & -3 & 0\\ 2 & 3 & 1 & 1 & 0\\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \overset{R_1\leftrightarrow R_2}{\Leftrightarrow} \begin{bmatrix} 1 & 0 & -1 & -3 & 0\\ 0 & 2 & 2 & 4 & 0\\ 2 & 3 & 1 & 1 & 0\\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \Leftrightarrow$$

$R_2/2,\ R_3-2R_1,\ R_4+2R_1$ $\qquad$ R_3-$3R_2$, $-R_2+R_4$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0\\ 0 & 1 & 1 & 2 & 0\\ 0 & 3 & 3 & 7 & 0\\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 & -3 & 0\\ 0 & 1 & 1 & 2 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \Leftrightarrow$$

R_1+3R_3, R_2-$2R_3$, R_4+10R_3

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System form is;

$$\begin{aligned} w-y &=0\\ x+y &=0\\ z&=0 \end{aligned}$$

leading entries are w, x, and z , free entry is y

let $y=t$

$w=y=t$

$x=-y=-t$

$z=0$

solution is $w=t, x=-t, y=t, z=0,$ where $t\in R,\quad t\neq 0.$

so there are inifitly many solutions.

Example:11. Solve the homogeneous system of linear equations

$$\begin{aligned}
x_1 + 3x_2 \qquad + x_4 &= 0 \\
x_1 + 4x_2 + 2x_3 \qquad &= 0 \\
-2x_2 - 2x_3 - x_4 &= 0 \\
2x_1 - 4x_2 + x_3 + x_4 &= 0 \\
x_1 - 2x_2 - x_3 + x_4 &= 0
\end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix} \underset{R_2-R_1,\ -R_3,\ R_4-2R_1,\ R_5-R_1}{\approx} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix}$$

$$\underset{R_3-R_2,\ R_4-10R_2,\ R5+5R_2}{\approx} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix} \underset{2R_4+21R_3,\ 2R_5+9R_3}{\approx} \begin{bmatrix} 1 & & 30 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 41 & 0 \\ 0 & 0 & 0 & 17 & 0 \end{bmatrix}$$

$$\underset{R_4/41,\ -R_5/17-R_4/41}{\approx} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1-R_4,\ R_2+R_4,\ (R_3-3R_4)/-2}{\approx} \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_1-2R_3}{\approx} \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1-3R_2}{\approx} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equation form is $x_1 = 0, x_2 = 0, x_3 = 0$ and $x_4 = 0$.

There is only trivial solution and there are no additional non-trivial solutions.

Example:12. For which value (s) of λ, the system of equations have non – trivial solutions,

$$\begin{aligned}(\lambda-3)x+\quad\quad y&=0\\ x+(\lambda-3)y&=0\end{aligned}$$

Solution:

The augmented matrix is $\begin{bmatrix}\lambda-3 & 1 & 0\\ 1 & \lambda-3 & 0\end{bmatrix}$

$$\approx\begin{bmatrix}1 & \lambda-3 & 0\\ \lambda-3 & 1 & 0\end{bmatrix}\approx\begin{bmatrix}1 & \lambda-3 & 0\\ 0 & 1-(\lambda-3)^2 & 0\end{bmatrix}$$

The equation form is

$$x+(\lambda-3)y=0 \qquad .1$$
$$[1-(\lambda-3)^2]y=0 \qquad .2$$

if $y = 0 \Rightarrow x = 0$, which are trivial solution,

hence $y \neq 0 \Rightarrow [1-(\lambda-3)^2]=0$

$$1-(\lambda^2-6\lambda+9)=0$$
$$1-\lambda^2+6\lambda-9=0$$
$$-\lambda^2+6\lambda-8=0$$
$$\lambda^2-6\lambda+8=0$$
$$(\lambda-2)(\lambda-4)=0$$

1. Both equations are identical for $\lambda = 2$ or $\lambda = 4 \Rightarrow$ there are infinitely many solutions.

2. If $\lambda = 4$, then $x + y = 0$, when $x = t$
   $y = -t$, where $t \in R$, $t \neq 0$.

3. If $\lambda = 2$, then $x - y = 0$, when $x = t$
   $y = t$, where $t \in R$, $t \neq 0$.

4. If $\lambda \neq 2$, and $\lambda \neq 4$, then the system will have trivial solution.

Example:13. For what values of λ the following system of equations

$$x+(\lambda-1)y=0$$
$$(\lambda-1)x+y=0$$

has (a) unique solution, and (b) infinitly many solutions.

Solution:

Augmented form of the homogeneous system

$$\begin{bmatrix} 1 & \lambda-1 & 0 \\ \lambda-1 & 1 & 0 \end{bmatrix}$$

Reducing it to reduced row echlon form,

multiply R_1 by $-(\lambda-1)$ and add to R_2

$$\approx\begin{bmatrix} 1 & \lambda-1 & 0 \\ 0 & 1-(\lambda-1)^2 & 0 \end{bmatrix}$$

From R_2, the system will have solution, if

$$\left[1-(\lambda-1)^2\right]y=0 \Rightarrow \lambda(2-\lambda)y=0.$$

If $y=0$, then $x=0$ and system will have unique trivial solution $x=0$, $y=0$.

If $y \neq 0$, then $\lambda(2-\lambda)=0 \Rightarrow \lambda=0$ or $\lambda=2$,

(a) if $\lambda=0$ the $x+y=0$ or $x=-y$, then the system will have infinite many solutions,

(b) if $\lambda=2$ the $x-y=0$ or $x=y$, then the system will have infinite many solutions, and

(c) if $\lambda \neq 0$ or $\lambda \neq 2$ then the system will have unique solution and that is trivail solution $x=0$, $y=0$.

EXERCISE 1.3

Homogenous system of equations

1. Solve the homogenous system of equations by reducing it to reduced row echelon form:

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 0 \\ 3x_1 + 2x_2 + 11x_3 &= 0 \end{aligned}$$

2. Solve the homogenous system of equations by reducing it to reduced row echelon form:

$$\begin{aligned} 2x - 9y + 3z + 2w &= 0 \\ x - 4y \qquad - w &= 0 \\ 2x - 6y - 2z + 5w &= 0. \end{aligned}$$

3. Solve the homogenous system of equations by reducing it to reduced row echelon form:

$$\begin{aligned} x + y + z &= 0 \\ x - y + 2z &= 0 \\ 3x - y + 5z &= 0 \end{aligned}$$

4. For what values of λ does the following system has non-trivial solution

$$\begin{aligned} x + 2y + 4z &= 0 \\ 3x - 2y + \lambda z &= 0 \\ 5x + 3y + z &= 0 \end{aligned}$$

5. For what values of λ does the following system has non-trivial solution

$$\begin{aligned} 3x + y - \lambda z &= 0 \\ 4x - 2y - 3z &= 0 \\ 2\lambda x + 4y + \lambda z &= 0 \end{aligned}$$

$Ans: \quad \lambda = 1, -9$

Rules for Good Listening

1. Be on time for class
2. Be ready to begin when class begins.
3. Be prepared to participate.
4. Be visible
5. Turn in assignments on time
6. Do not expect extra help if you have cut class.
7. Do not try to be the class clown.
8. Do not start getting ready to leave until the teacher is finished.
9. Do not ask questions that annoy teachers.

NOTE TAKING

1. First Step - PREPARATION

Use a large, loose-leaf notebook. Use only one side of the paper. (you then can lay your notes out to see the direction of a lecture.) Draw a vertical line 2 1/2 inches from the left side of you paper. This is the recall column. Notes will be taken to the right of this margin. Later key words or phrases can be written in the recall column.

2. Second Step - DURING THE LECTURE

Record notes in paragraph form. Capture general ideas, not illustrative ideas. Skip lines to show end of ideas or thoughts. Using abbreviations will save time. Write legibly.

3. Third Step - AFTER THE LECTURE

Read through your notes and make it more legible if necessary. Now use the column. Jot down ideas or key words which give you the idea of the lecture. (REDUCE) You will have to reread the lecturer's ideas and reflect in your own words. Cover up the right-hand portion of your notes and recite the general ideas and concepts of the lecture. Overlap your notes showing only recall columns and you have your review.

STEPS IN THE SQ3R METHOD

1.SURVEY

- **the material for content and organization.**
- **Major headings for each section.**
- **Topic sentences, including statements in italics or boldface type.**
- **Pictures, charts, diagrams, maps, etc.**

2.QUESTIONS

- **based on the survey help the reader's thinking.**
- **Ask questions based on the headings of each section.**
- **Use any study guides provided by the teacher or author.**

3.READ **to answer your questions.**

4.RECITE

- **to check comprehension and to utilize information.**
- **Write brief summary statements to questions.**
- **Discuss ideas orally in class.**
- **Organize major points in outline form.**
- **Utilize study techniques such as underlining key sentences, making brief notes, writing in the margin, etc.**

5.REVIEW

- **for immediate and delayed recall.**
- **Reread notes to recall major points.**
- **Use quizzes to study for major tests.**
- **Utilize skimming and scanning techniques to refresh memory.**

CHAPTER 2

Matrices

2.1 Properties of a matrix

1. **Transpose of a Matrix:** A transpose of a matrix is obtained by interchanging rows and corresponding columns of the given matrix. The transpose of the matrix A is denoted A^t.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of the Transpose of a matrix

1. $(A^t)^t = A$
2. $(AB)^t = B^t A^t$
3. $(kA)^t = kA^t$, where k is a scalar.
4. $(A+B)^t = A^t + B^t$

2. **Symmetric Matrix:** A square matrix is symmetric if $A^t = A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = A$$

3. **Skew – symmetric Matrix :** A square matrix is skew symmetric if $A^t = -A$.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, \quad A^t = -A.$$

4. **Equality of matrix:** Two matrices are equal, if these of same size and corresponding entries are equal.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

A and B are equal matrices when these of the same size and corresponding entries are equal.

8. Unit Matrix: A diagonal matrix with all diagonal entries are unity '1'

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Transpose of a Matrix: A transpose of a matrix is obtained by interchanging rows and corresponding columns of the given matrix. The transpose of the matrix A is denoted A^t.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of the Transpose of a matrix

1. $(A^t)^t = A$
2. $(AB)^t = B^tA^t$
3. $(kA)^t = kA^t$, where k is a scalar.
4. $(A+B)^t = A^t + B^t$

10. **Symmetric Matrix:** A square matrix is symmetric if $A^t = A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = A$$

11. **Skew – symmetric Matrix :** A square matrix is skew symmetric if $A^t = -A$.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}, \quad A^t = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, \quad A^t = -A.$$

12. **Equality of matrix:** Two matrices are equal, if these of same size and corresponding entries are equal.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$ A and B are equal matrices when these of the same size and corresponding entries are equal.

Example:1. Write down the system of equation, if matrices A and B are equal

$$A = \begin{bmatrix} x-2 & y-3 \\ x+y & z+3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3+z \\ z & y \end{bmatrix}$$

Solution: A and B are of the same size, hence

$$A = B \Rightarrow$$

$$\begin{aligned} x-2 &= 1 \\ y-3 &= 3+z \\ x+y &= z \\ z+3 &= y \end{aligned}$$

System of equations are

$$\begin{aligned} x \qquad &= 3 \\ y - z &= 6 \\ x + y - z &= 0 \\ -y + z &= -3 \end{aligned}$$

13. Addition of matrices:

Matrices of the equal size can be added entry wise.

Example:2. Find A + B where $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & -5 \\ 3 & 4 \end{bmatrix}$

Solution:

$$A + B = \begin{bmatrix} 2+1 & 1-1 \\ 3+2 & 4-5 \\ 4+3 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & -1 \\ 7 & 9 \end{bmatrix}$$

2.2 <u>**Scalar Multiplication:**</u>

If a matrix is multiplied by a scalar α, then each entry is multiplied by scalar α.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 0 \\ 2 & 2 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 3 & 0 \\ 3 & 3 & 6 \end{bmatrix}$$

2.3 **Matrix Multiplication:**

The product of two matrices A and B is possible if the number of columns of A is equal to number of rows in B, the method is being explained by following example:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{2x3}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}_{3X4}$$

$$\begin{array}{ccc} A \text{ x} & B = & C \\ 2x3 & 3x4 & 2x4 \end{array}$$

$$AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$$

c_{11} = 1x4 + 2x0 + 4x2 = 4 + 0 + 8 = 12
c_{12} = 1x1 + 2x(-1) + 4x7 = 1 - 2 + 28 = 27
c_{13} = 1x4 + 2x3 + 4x5 = 4 + 6 + 20 = 30
c_{14} = 1x3 + 2x1 + 4x2 = 3 + 2 + 8 = 13
c_{21} = 2x4 + 6x0 + 0x2 = 8 + 0 + 0 = 8
c_{22} = 2x1 + 6x(-1) + 0x7 = 2 -6 + 0 = -4
c_{23} = 2x4 + 6x3 + 0x5 = 8 + 18 + 0 = 26
c_{24} = 2x3 + 6x1 + 0x2 = 6 + 6 + 0 = 12

$$\text{AB} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

NOTE: AB $\neq$ BA

2.4 Inverse of a 2x2 matrix

Consider a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If ad – bc $\neq 0$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:3. Find inverse of matrix $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

3x5 –2x4 =15 – 8 = 7

$A^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$.

Properties of Inverse

1. $A^{-1}A = A\,A^{-1} = I$
2. If A and B are invertible matrices of the same size , then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

2.5 Power of a matrix

1. $A^0 = I$
2. $A^n = A.A.A\ldots A$ (n-factors), where n>0.
3. $\mathbf{A^{-n}} = (A^{-1})^n = A^{-1}.A^{-1}.A^{-1}....A^{-1}$ (n- factors) , where n>0.
4. $A^rA^s = A^{r+s}$
5. $(A^r)^s = A^{rs}$
6. $(A^{-1})^{-1} = A$
7. $(A^n)^{-1} = (A^{-1})^n$, n = 0,1,2,...
8. $(kA)^{-1} = \dfrac{1}{k}A^{-1}$, where k is a scalar.

Example:4. Let A be an invertible matrix and suppose that inverse of 7A is $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$, find matrix A

Solution: $(7A)^{-1} = \frac{1}{7}A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$

$$A^{-1} = 7\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -14 & 49 \\ 7 & -21 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = -\frac{1}{49}\begin{bmatrix} -21 & -49 \\ -7 & -14 \end{bmatrix} = \frac{7}{49}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}.$$

Example:5. Let A be a matrix $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ compute A^3, A^{-3}, $A^2 - 2A + I$.

Solution:

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^{-3} = (A^3)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

$$A^2 - 2A + I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

Example:6. Find inverse of the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Solution:

$$ad - bc = \cos^2\theta + \sin^2\theta = 1,$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

EXERCISE 2

MATRICES AND MATRIX OPERATIONS

1. If A is 2x2 matrix and $B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, if $AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, find A.

2. Let $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix}$, find A^2+3A.

3. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$, find A^3-6A+11A-$6I_3$

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x & 3 \end{bmatrix}$, $C = \begin{bmatrix} y \\ 1 \\ z \end{bmatrix}$ and $AB^T = C$, find values of x, y and z.

5. Given the matrices $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$ and $X = \begin{bmatrix} x & y & z \end{bmatrix}$

   solve the following equation $AX^T = 3X^T + 2B^T$ for x, y and z.

6. Find the traspose of AB if $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

7. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ find if possible, a matrix B such that AB = C

8. Find if $(5A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

9. Find all values of a, b, and c for which A is symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

10. Find all values of a and b for which A and B are invertible

$$A = \begin{bmatrix} a+b-1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 2a-3b-7 \end{bmatrix}$$

11. Find the product of transpose with its matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$$

12. Find A if $A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix}$

2.6 Elementary Matrix

An nxn matrix is called *elementary matrix*, if it can be obtained from nxn identity matrix by performing a single row operation.

Examples: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3x3 identity matrix.

Elementary matrices E_1,E_2 and E_3 can be obtained by single row operation.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{-3}R_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \text{-2}R_3+R_2$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

NOTE:

When a matrix A is multiplied from the left by an elementary matrices E, the effect is same as to perform an elementary row operation on A.

Example: 1.

Let A be a 3x4 matrix, $A=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}$ and

E be 3x3 elementary matrix obtained by row operation $3R_1+R_3$ from an Identity matrix

$$E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}$$

$$EA=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 4 & 4 & 10 & 9\end{bmatrix}, 3R_1+R_3 .$$

2.7 Method for finding Inverse of a matrix

To find an inverse of matrix A, we perform a sequence of elementary row operations that reduce

$$[A \mid I] \text{ to } [I \mid A^{-1}]$$

Example:2. Find inverse of a matrix $A=\begin{bmatrix}1 & 4\\ 2 & 7\end{bmatrix}$ by using Elementary matrix method.

Solution:

$$[A|I]=\left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 2 & 7 & 0 & 1\end{array}\right]$$

$$\approx\left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 0 & -1 & -2 & 1\end{array}\right] \quad -2R_1+R_2$$

$$\approx\left[\begin{array}{cc|cc}1 & 4 & 1 & 0\\ 0 & 1 & 2 & -1\end{array}\right] \quad -R_2$$

$$\approx\left[\begin{array}{cc|cc}1 & 0 & -7 & 4\\ 0 & 1 & 2 & -1\end{array}\right] \quad -4R_2+R_1$$

$$=[I|A^{-1}]$$

$$A^{-1}=\begin{bmatrix}-7 & 4\\ 2 & -1\end{bmatrix}$$

Example:4. Use Elementary matrix method to find inverses of

$$A=\begin{bmatrix}3&4&-1\\1&0&3\\2&5&-4\end{bmatrix} \qquad \text{if A is invertible.}$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc}3&4&-1&1&0&0\\1&0&3&0&1&0\\2&5&-4&0&0&1\end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\3&4&-1&1&0&0\\2&5&-4&0&0&1\end{array}\right] R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&4&-10&1&-3&0\\0&5&-10&0&-2&1\end{array}\right] -3R_1+R_2, -2R_1+R_3$$

$$\approx \left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&4&-10&1&-3&0\\0&1&0&-1&-2&1\end{array}\right] -R_2+R_3$$

$$\approx \left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&1&0&-1&1&1\\0&0&-1&\frac{1}{2}&\frac{-7}{10}&\frac{-2}{5}\end{array}\right] R_2 \leftrightarrow R_3, \frac{(-4R_3+R_2)}{10}$$

$$\approx \left[\begin{array}{ccc|ccc}1&0&0&\frac{3}{2}&\frac{-11}{10}&\frac{-6}{5}\\0&1&0&-1&1&1\\0&0&1&\frac{-1}{2}&\frac{7}{10}&\frac{2}{5}\end{array}\right] -3R_3+R_1, -R_3$$

$$\approx [I|A^{-1}]$$

$$A^{-1}=\begin{bmatrix}\frac{3}{2}&\frac{-11}{10}&\frac{-6}{5}\\-1&1&1\\\frac{-1}{2}&\frac{7}{10}&\frac{2}{5}\end{bmatrix}.$$

2.8 Solving Linear system by Inverse Matrix

Let a given linear system of equations is

$$AX = B$$

Find A^{-1}

Multiply with A^{-1} from left

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B \text{ is a solution.}$$

Note: To find A^{-1} we use *Elementary Matrix method.*

Example:5.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 3$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of } AX = B$$

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\2&2&1&0&1&0\\2&3&1&0&0&1\end{array}\right]$$

$$\approx\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&-4&-1&-2&1&0\\0&-3&-1&-2&0&1\end{array}\right]\quad \mathrm{R}_2-2R_1, R_3-2R_1$$

$$\approx\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&-4&-1&-2&1&0\\0&0&-1&-2&-3&4\end{array}\right]\quad 4R_3-3R_2$$

$$\approx\left[\begin{array}{ccc|ccc}1&3&0&-1&-3&4\\0&-4&0&0&4&-4\\0&0&-1&-2&-3&4\end{array}\right]\quad R_1+R_3,\ R_2-R_3$$

$$\approx\left[\begin{array}{ccc|ccc}1&3&0&-1&-3&4\\0&1&0&0&-1&1\\0&0&-1&-2&-3&4\end{array}\right]\quad -\tfrac{1}{4}R_2$$

$$\approx\left[\begin{array}{ccc|ccc}1&3&0&-1&-3&4\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right]\quad -R_3$$

$$\approx\left[\begin{array}{ccc|ccc}1&0&0&-1&0&1\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right]\quad -3R_2+R_3$$

$$\equiv\ \left[\,\mathrm{I}\,\middle|\,\mathrm{A}^{-1}\right]$$

$$\mathrm{A}^{-1}=\begin{bmatrix}-1&0&1\\0&-1&1\\2&3&-4\end{bmatrix}$$

$$\mathrm{X}=\mathrm{A}^{-1}\mathrm{B}=\begin{bmatrix}-1&0&1\\0&-1&1\\2&3&-4\end{bmatrix}\begin{bmatrix}4\\-1\\3\end{bmatrix}=\begin{bmatrix}-1\\4\\-7\end{bmatrix}$$

Solution set is $x_1=-1, x_2=4, x_3=-7.$

Example:6.

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$x + z = -y + 4$$
$$x + 2z = y + 1$$
$$y + z = 3$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$ is in form of AX = B

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 - R_1$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 & 1 & 2 \end{array}\right] \quad R_1 - R_2,\ 2R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right] \quad \frac{1}{3}R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right] \quad -R_3 + R_2$$

$$\equiv \left[I \,|A^{-1} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 1 & 0 & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution set is $x = 1, y = 2, y = 1$.

EXERCISE 3

SOLUTION OF SYSTEM OF EQUATION BY FINDING INVERSE OF MATRIX BY ELEMENTARY ROW OPERATIONS

1. Use elementary matrix method (using row operations) to find A^{-1} of the matrix

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

2. Use elementary matrix method to find A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

3. Use elementary matrix method to find A^{-1} of the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$$

4. Solve the matrix equation by finding A^{-1}, by using the Elementary Row Operation (Row Reduction)

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

In problems 3 to 5, write the following system of equations in matrix equation and solve by using row operations (elementary matrix method) for find the inverse of the coefficients matrix

5.
$$\begin{aligned} 2x + 4y + 3z &= 6 \\ y - z &= -4 \\ 3x + 3y + 7z &= 7 \end{aligned}$$

4.
$$\begin{aligned} -1x + z &= 2 \\ -5x + y + 3z &= -2 \\ 7x - y - 4z &= 1 \end{aligned}$$

5. $$\begin{aligned} 2x + 2y + 3z &= 8 \\ y - z &= -6 \\ 3x + 3y + 7z &= 6 \end{aligned}$$

6. Solve $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 9 \end{bmatrix}$

7. Find A^{-1} by using [A: I] method for $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

8. If A is a 2x3 matrix, $B = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$, if $AB = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, find A.

9. If $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ find (a) B^{-1} (b) B^3

EFFECTIVE LISTENING

1. Desire to become a better listener.
2. Stop talking.
3. Look at the teacher.
4. Leave your emotions behind.
5. Get rid of distractions.
6. Get the main points.
7. Don't argue mentally.
8. Listen for what is not said.
9. Avoid jumping to conclusions.
10. Avoid hasty judgments.

Elements of Study

1. Time Management

2. Better concentration

3. Listening and Note taking

4. Reading

5. Preparing for examination

6. Writing examination

20 TIME MANAGEMENT TECHNIQUES

STUDY WHEN:

1. Plan two study hours for every hour you spend in class.
2. Study difficult (or boring) subjects first.
3. Avoid scheduling marathon study sessions.
4. Be aware of your best time of day.
5. Use waiting time.
6. Use a regular study area.

STUDY WHERE:

1. Choose a place that minimizes visual and auditory distractions.
2. Use your bed room or study room.
3. Don't get too comfortable. Sit (or even stand) so that you can remain awake and attentive.
4. Find a better place when productivity falls off.

YOU AND THE OUTSIDE WORLD:

1. Pay attention to your attention.
2. Avoid noise distractions.
3. Notice how others misuse your time.
4. Get off the phone.
5. Learn to say no.
6. Hang a "Do Not Disturb!" sign on your door.
7. Ask: "What is one task I can accomplish toward my goal?"
8. Ask: "How did I just waste time?"
9. Ask: "Would I pay myself for what I'm doing right now?"
10. Ask: "Can I do just one more thing?" (Stretch yourself).

CHAPTER 3

Determinants

3.1 Determinant of a Matrix

A number is associated with each matrix that is referred as determinant of matrix A is denoted by $|A|$ or det(A)

3.2 Evaluating determinant by direct multiplication

The determinant of a 2x2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ is}$$

$$\text{Det (A)} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \qquad a_{11}a_{22} - a_{21}a_{12}$$

The determinant of 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is}$$

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{23} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{matrix}$$

Note: This method does not work for 4 x 4 or higher order.

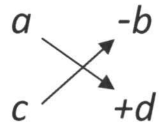
3.1 Determinant of a matrix

A number us associated with each matrix A that is referred as determinant of matrix A and is denoted by | A | or det (A)

3.2 Evaluating determinant of Matrix

For 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



$$|A| = \det A = ad - cb$$

For 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}
\det A &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - (a_{12}a_{21}a_{33} - a_{12}a_{31}a_{23}) + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}
\end{aligned}$$

Example .1: Find the determinant of matrics

$$A=\begin{bmatrix}2 & 3\\ 5 & 9\end{bmatrix}, \quad B=\begin{bmatrix}2 & 4 & 5\\ 3 & 6 & 8\\ 4 & 5 & 9\end{bmatrix}$$

Solution: Determinant of 2x2 matrix

$$A=\begin{bmatrix}2 & 3\\ 5 & 9\end{bmatrix} \qquad \det A = 2x9 - 3x5 = 18 - 15 = 3.$$

Determinant of 3x3 matrix

$$B=\begin{bmatrix}2 & 4 & 5\\ 3 & 6 & 8\\ 4 & 5 & 9\end{bmatrix}$$

$$\begin{aligned}\det(B) &= 2\begin{vmatrix}6 & 8\\ 5 & 9\end{vmatrix} - 4\begin{vmatrix}3 & 8\\ 4 & 9\end{vmatrix} + 5\begin{vmatrix}3 & 6\\ 4 & 5\end{vmatrix}\\ &= 2(54-40) - 4(27-32) + 5(15-24)\\ &= 2(14) - 4(-5) + 5(-9)\\ &= 28 + 20 - 45\\ &= 48 - 45\\ &= 3\end{aligned}$$

Note: We can write it as det(A) or $|A|$ or $\begin{vmatrix}2 & 3\\ 5 & 9\end{vmatrix}$ or $\det\begin{bmatrix}2 & 3\\ 5 & 9\end{bmatrix}$

3.3 Finding determinant by method of co-factors

Minor The minor of an element of a matrix a_{ij} of a matrix A, denoted by M_{ij}, is the determinant of the matrix obtained by deleting the row and column containing a_{ij}.

$$\text{Example: } A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

The minor M_{23} of the element a_{23} of matrix

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the determinant of 2x2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$. Thus

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}.$$

Cofactor of an element a_{ij} of a matrix A, denoted by C_{ij}, is defined as

$\mathbf{C_{ij} = (-1)^{i+j} M_{ij}}$, where M_{ij} is minor of the element a_{ij}.

NOTE: $C_{ij} = M_{ij}$ when $i + j$ is even, and $C_{ij} = -M_{ij}$ when $i + j$ is odd.

Determinant of matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by method of cofactor is

$$\text{Det}(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

NOTE: Determinant of a matrix can be obtained by expanding about any column or row.

Finding determinant by expanding it through second row:

$$\text{Det}(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

Example:2. Find the determinant of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} \text{Det}(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (1)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - (2)\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} + (3)\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} \\ &= (1)(-5) - (2)(-6) + (3)(2) \\ &= -5 + 12 + 6 \\ &= 13 \end{aligned}$$

Example:3. Find determinant of matrix if $A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$

Solution: Expanding from 4^{th} row

$$\text{Det (A)} = -(1)\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 2 & 5 \\ 2 & 2 & 3 \\ 3 & 1 & 5 \end{vmatrix} - (4)\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 3 \\ 3 & 2 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 1 & 2 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= -(1)(4) + (0)(?) - (4)(34) + (0)(?)$$

$$= -4 - 136 = -140.$$

Example : 4. Find all values of λ for which $\det(A) = 0$ for matrix

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Solution: $\det(A) = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & 2 \\ 0 & \lambda - 1 \end{vmatrix} + (0)\begin{vmatrix} 0 & \lambda \\ 0 & 3 \end{vmatrix}$

$= (\lambda-4)\,[\lambda(\lambda-1) - 6]$

$= (\lambda-4)\,[\lambda^2 - \lambda - 6]$

$= (\lambda-4)(\lambda-3)(\lambda+2)$

$\det(A) = 0.$

$(\lambda-4)(\lambda-3)(\lambda+2) = 0.$

$\Rightarrow \lambda = 4,\ \lambda = 3,\ \lambda = -2.$

3.4 Evaluating Determinant by row operations

1. If matrix A_1 is obtained from matrix A by the interchange of two rows , then **$\det(A_1) = -\det(A)$.**

2. If matrix A_2 is obtained from matrix A by the multiplication of a row of A by a constant k , then **$\det(A_2) = k\det(A)$.**

3. If matrix A_3 is obtained from the matrix A by addition of a multiple of one row to another row , then **$\det(A_3) = \det(A)$.**

Example:5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$, and $\det(A) = 2$. Find determinant of

(i)$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix}$, (ii) $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, (iii) $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution: (i) A_1 is obtained from A by interchanging R_2 and R_3 of A,
$\det(A_1) = -\det(A) = -2$.

(ii) A_2 is obtained from A by multiplying R_3 of A by $\frac{1}{2}$,
$\det(A_2) = \frac{1}{2}\det(A) = \frac{1}{2}(2) = 1$.

(iii) A_3 is obtained by row operation $-2R_2+R_1$,
$\det(A_3) = \det(A) = 2$.

NOTE:

1. If A is any square matrix that contains a row of zeros, then $\det(A) = 0$.
2. If a square matrix has two proportional rows, then $\det(A) = 0$.
3. In case of upper or lower triangular matrix, determinant is the product of the diagonal elements.

Lower **triangular matrix**

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \det(A) = a_{11}a_{22}a_{33}$$

Upper **triangular matrix**

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad \det(B) = a_{11}a_{22}a_{33}$$

Diagonal matrix

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Example:6.

Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6,$ find (a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$, (b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$, (d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$.

Solution:

(a) $$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\underset{R_1 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}} = (-1)(-1)\underset{R_2 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-1)(-1)(6) = 6$$

(b) $$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(6) = -72$$

(c) $$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \underset{R_1 - R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = 6$$

(d) $$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = (-3)\underset{4R_2 + R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-3)(6) = -18$$

Example:7. Evaluate the determinant by row reduction

$$\text{Det A} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

Solution:

$$\det A = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \qquad 2R_1 + R_2, -2R_3 + R_4$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \qquad -R_4 + R_5$$

$$= (1)(-1)(1)(1)(2) = -2$$

Example:8. Find the value(s) of x if det A = -12, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-3 & -3 \\ 1 & x-4 & 0 \end{bmatrix}$$

Solution: Performing row operations $-2R_1+R_2$, $-R_1+R_3$

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x-3 & -3 \\ 0 & x-4 & 0 \end{vmatrix} = (1)\begin{vmatrix} x-3 & -3 \\ x-4 & 0 \end{vmatrix} - (0) + (0)$$

$$= 3(x-4)$$

$$\det A = -12 \quad \Rightarrow -3x - 12 = -12$$
$$-3x = 0$$
$$x = 0.//$$

NOTE: Operations on columns are same as on rows.

Theorem:

For an nxn matrix A, following are equivalent:

1. $\det(A) \neq 0$,
2. A^{-1} exists, and
3. $AX = B$ has a unique solution for any B.
4. A is invertible

3.5 Properties of Determinantial Function

1. **If A is a nxn matric $\det(kA) = k^n \det(A)$,**
2. **$\det(A + B) \neq \det(A) + \det(B)$,**
3. **$\det(AB) = \det(A).\det(B)$,**
4. $\det(A^{-1}) = \dfrac{1}{\det A}$,
5. **A square matrix is invertible if and only if $\det(A) \neq 0$,and**
6. **$\det(A^t) = \det(A)$**
7. If $\det A = 0$, then matrix A is singular matrix
8. $AX = 0$, will have non-trivial solution iff $\det A = 0$

Example :9. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = -7$ find

(a) det (3A) , (b) $\det(2A)^{-1}$, (c) $\det(2A^{-1})$ and (d) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

Solution: a. $\det(3A) = 3^3 \det A = 27(-7) = -189$

b. $\det(2A)^{-1} = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3 \det(A)} = \dfrac{1}{8(-7)} = \dfrac{-1}{56}$

c. $\det(2A^{-1}) = 2^3 \det(A) = \dfrac{2^3}{\det(A)} = \dfrac{8}{-7} = \dfrac{-8}{7}$

d. $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}^t = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7) = 7$

Example:10. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Solution.

det(A) = det (A^t)

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$R_2 - R_1, R_3 - R_1$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$R_3 - R_2$

$$= (b-c)(c-a)(c-b)$$

Example:11. Without directly evaluating using properties of determinant show that

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Solution:

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} R_1 + R_2$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0.$$

EXERCISE 4

DETERMINANTS AND PROPERTIES OF DETERMINANT

In problems 1 to 8 evaluate the determinant by the definition of a determinant or by direct multiplication.

1. $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

2. $\begin{vmatrix} 5 & -6 \\ 7 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 6 \\ 5 & 4 \end{vmatrix}$

4. $\begin{vmatrix} 3 & 6 \\ 4 & 8 \end{vmatrix}$

5. $\begin{vmatrix} 2 & -5 & -1 \\ -1 & 6 & 1 \\ -1 & -3 & -3 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & -4 \\ 4 & 3 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \\ -6 & 7 & 2 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 5 & -1 \\ 4 & 1 & 0 \\ 2 & -3 & 0 \end{vmatrix}$

In problems 9 to 16 evaluate the determinant by using row operations to introduce zeros,

9. $\begin{vmatrix} 8 & -1 & -1 \\ -1 & 7 & -2 \\ 1 & 1 & -6 \end{vmatrix}$

10. $\begin{vmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 4 & 2 & 1 & 1 \end{vmatrix}$

11. $\begin{vmatrix} 2 & 3 & -3 & 8 \\ 4 & 0 & 1 & 5 \\ 9 & 0 & -5 & 9 \\ 2 & 0 & 12 & 8 \end{vmatrix}$

12. $\begin{vmatrix} 3 & 1 & 2 & 6 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 5 & 10 \\ 1 & 2 & 1 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 4 & 5 \\ 2 & 5 & 6 & 7 \\ 3 & 2 & 4 & 5 \end{vmatrix}$

14. $\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$

15. $\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 3 & 8 & 1 \\ 0 & 2 & 3 & 4 \end{vmatrix}$

16. $\begin{vmatrix} 1 & 4 & 1 & 2 & 3 \\ -3 & -4 & -3 & -5 & -8 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & -4 & 1 \\ 0 & 0 & 0 & 3 & 6 \end{vmatrix}$

17. If $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \\ 5 & -2 & -3 \end{bmatrix}$, find (i) det A (ii) det A^T

18. Find values of λ the determinant of the matrix

$\begin{bmatrix} \lambda^2 & 4 & 1 \\ -4 & -\lambda & 2 \\ 6 & 3 & \lambda^2 \end{bmatrix}$, if the inverse of matrix $\begin{bmatrix} \lambda^2 & 1 \\ 1 & \lambda \end{bmatrix}$

does not exist.

19. For what values of λ, A is not invertible.

(i) $A = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & \lambda \end{bmatrix}$

20. Find the values of λ for which the following matrix does not have an inverse

(i) $\begin{bmatrix} 1 & \lambda - 2 & 0 \\ 1 & 2\lambda & 0 \\ 0 & 2 & \lambda \end{bmatrix}$

(ii) $\begin{bmatrix} -\lambda & \lambda-1 & \lambda+1 \\ 1 & 2 & 3 \\ 2-\lambda & \lambda+3 & \lambda+7 \end{bmatrix}$

20. Find the values of λ for which an inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2\lambda & 0 \\ 0 & 2 & \lambda \end{bmatrix} \text{ exists.}$$

21. Find the value of x if det A = -12, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & x-3 & 1 \\ 1 & x-4 & 2 \end{bmatrix}$$

22. Solve the determinantal equation

$$\begin{vmatrix} x & -3 & 2 \\ 2 & x & -3 \\ -3 & 2 & x \end{vmatrix} = -19$$

23. Use properties of the determinant to evaluate

$$\det(2A) + \det(2A^{-1}) + 2\det A \,.\, \det A^{-1},$$

$$\text{if } A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & -3 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

24. Use properties of the determinant to prove

$$\begin{bmatrix} 2x_1+2y & 2x_1-2y_1 & z_1 \\ 2x_2+2y_2 & 2x_2-2y_2 & z_2 \\ 2x_3+2y_3 & 2x_3-2y_3 & z_2 \end{bmatrix} = -8\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

25. Show that inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ c+b & c+a & a+b \end{bmatrix}$ does not exist.

26. If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0,$

then prove that $xyz = -1$, where x, y, z are unequal.

27. Prove that $\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$

28. Use properties of the determinant to evaluate

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ a & b & c & d \\ bcd & acd & abd & abc \end{vmatrix}$$

29. Use properties of the determinant to show that

$$\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix} = 5\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}$$

30. Find the value of det[$(2A)^2 (A^{-1})^2$], where

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \text{ and } \qquad \det A = 3.$$

31. Without expanding the determinant show that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix} = (b-c)(c-a)(c-b)(a+b+c)$$

32. Find the values of x for which the matrix

$$\begin{bmatrix} 2x-1 & 2-x & x \\ x & 1 & x \\ 2x & 1+x & 1+x \end{bmatrix} \text{ has no inverse.}$$

33. For what values of x matrix A is singular

$$A = \begin{bmatrix} x & -1 & 0 \\ 0 & x-3 & -2 \\ 2 & 1 & x+2 \end{bmatrix}$$

34. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and detA = 5, find

(a) $\det [(2A)^{-1}]$ (b) $\det (2A)$

(c) $\det (3A^{-1}) + \det [(3A)^{-1}] + \det (2A)$

(d) $\begin{bmatrix} -a & -b & -c \\ 3a+d & 3b+e & 3c+f \\ 4g & 4h & 4i \end{bmatrix}$

35. If $\begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix} = 3$ evaluate $\begin{vmatrix} a+x & b+y & c+z \\ l-x & m-y & n-z \\ a+x+l & b+y+m & c+z+n \end{vmatrix}$

36. Find the matrix A if $(5A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

37. If A is 3x3 matrix and if $\det(A) = 10$, find

(a) $\det(7A)$ (b) $\det (5A^{-1})$

38. Find det (A) if (i) $\det(A^{-1}) = 5$
(iii) $\det(2A) = 25$ where A is 3x3 matrix.

39. Find non – trivial solutions, if any, of

$$(-4I_3 - A)\, x = 0 \text{ where } A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix} \text{ and } x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

40. Find the values of " a" for which an inverse of the matrix A exists

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a^2 \end{bmatrix}$$

41. Find the values of "a" for which the matrix A does not have an inverse

(i) $$A = \begin{bmatrix} 1 & a-2 & 0 \\ 1 & 2-a & 0 \\ 0 & 2 & a \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} -a & a-1 & a+1 \\ 1 & 2 & 3 \\ 2-a & a+3 & a+7 \end{bmatrix}$$

42. For what values of "λ", matrix A is not invertible

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & \lambda \end{bmatrix}$$

43. **If** $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ **and show that** $\det(A^{-1}) = \dfrac{1}{\det(A)}$

44. Solve by inspection

(i)

$$\begin{vmatrix} x & 5 & 7 \\ 0 & x+1 & 6 \\ 0 & 0 & 2x+1 \end{vmatrix} = 0$$

$$Ans: x = 0, x = -1, x = -\frac{1}{2}$$

(ii)

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & -3 & 9 \end{vmatrix} = 0$$

$$Ans: x = 1, x = -3$$

45. Solve

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$$

Ans: $x = -1$, $x = -1$, $x = -2$

3.6 Minors and cofactors of a Matrix

$$\text{Let A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor of $a_{ij} \equiv M_{ij}$,

is determinant obtained by deleting ith row and jth column.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ is determinant obtained by deleting 1st row and 1st column}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\boxed{\textbf{Cofactor of } a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}}$$

Signs of Cofactors

$$\text{For 2x2 – matrix} \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$\text{For 3x3 – matrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\text{For 4x4 – matrix} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Example:1. Find all minors and cofactors of the matrix

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{bmatrix}$$

Solution:

$$M_{11}=\begin{vmatrix}0 & 3\\ 5 & -4\end{vmatrix}=-15, M_{12}=\begin{vmatrix}1 & 3\\ 2 & -4\end{vmatrix}=-10, M_{13}=\begin{vmatrix}1 & 0\\ 2 & -5\end{vmatrix}=5$$

$$M_{21}=\begin{vmatrix}4 & -1\\ 5 & -4\end{vmatrix}=-11, M_{22}=\begin{vmatrix}3 & -1\\ 2 & -4\end{vmatrix}=-10, M_{23}=\begin{vmatrix}3 & 4\\ 2 & 5\end{vmatrix}=7$$

$$M_{31}=\begin{vmatrix}4 & -1\\ 0 & 3\end{vmatrix}=12,\quad M_{32}=\begin{vmatrix}3 & -1\\ 1 & 0\end{vmatrix}=10,\quad M_{33}=\begin{vmatrix}3 & 4\\ 1 & 0\end{vmatrix}=-4$$

$$\boxed{\textbf{Cofactor of } a_{ij}=C_{ij}=(-1)^{i+j}M_{ij}}$$

$$\begin{array}{lll} C_{11}=-15, & C_{12}=10, & C_{13}=5\\ C_{21}=11, & C_{22}=-10, & C_{23}=-7\\ C_{31}=12, & C_{32}=-10, & C_{33}=-4\end{array}$$

NOTE: Matrix of cofactors , $C=\begin{bmatrix}-15 & 10 & 5\\ 11 & -10 & -7\\ 12 & -10 & -4\end{bmatrix}$

<u>NOTE: Determinant of matrix of Cofactors by the method of Cofactors</u>

$$\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$
$$\det(A)=a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23}$$
$$\det(A)=a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}$$

The above equations can be used to check that the cofactors are found correctly as the values of determinants found must be equal, we open matrix from any row or column.

Example: 2 . Find the determinant of the matrix A by method of cofactors,

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution: : Using the cofactors found in the last example

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3(-15)+4(10)+(-1)(5) \\ &= -45 + 40 - 5 = -10 \end{aligned}$$

NOTE: 3. We can find determinant by opening matrix from second or third row or first column, the value of the determinant will be same

$$\begin{aligned} \det(A) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= (1)(11)+0(-10)+3(-7) = 11 - 21 = -10 \end{aligned}$$

$$\begin{aligned} \det(A) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= 2(12) +5(-10)+(-4)(-4) = 24 - 50 + 16 = -10 \end{aligned}$$

NOTE : 4. Determinant of A can be obtained by multiplying any row or any column of matrix A with the corresponding row or column of the matrix of cofactors.

NOTE: 5. Determinant of matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\textbf{det A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

3.7 . Inverse by method of Cofactors:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det \mathbf{A} \neq 0.$$

Step:1. **Find Matrix of cofactors**

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. **Find Adjoint of matrix A , adj(A)**

$$\mathrm{Adj}(\mathrm{A}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A) $\neq$ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

Example: 3 . Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix} \text{ by the method of cofactors.}$$

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \; C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\text{Matrix of cofactors,} \quad C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$$

$$\text{Adjoint of matrix A,} \; \text{adj(A)} = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-12)+0(-4)+3(6) \\ &= -24 + 18 = -6 \neq 0 \end{aligned}$$

Inverse of matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

NOTE :

If we can find A^{-1}, then solution of linear system $AX = B$ is $\quad X = A^{-1}B$

EXERCISE 5

FINDING INVERSE OF A MATRIX and SOLUTION OF SYSYTEM OF EQUATIONS

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ be a matrix. Find adjA and hence find A^{-1} if exists.

2. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ be a matrix. Find adjA and hence find A^{-1} if exists.

3. Consider the following system of equations

$$\begin{aligned} x - y + 3z &= 8 \\ 4x + y + 6z &= 24 \\ x \qquad - z &= -2 \end{aligned}$$

(a) write the system in the form AX = B,
(b) Find A^{-1},if exists,
(c) Use (a) and (b) to solve the given system.

4. Consider the following system of equations

$$\begin{aligned} x \qquad - z &= 2 \\ y + z &= 3 \\ 3x \qquad - 2z &= 5 \end{aligned}$$

(a) write the system in the form AX = B,
(b) Find A^{-1}, if exists,
(c) Use (a) and (b) to solve the given system.

5. Let A be the coefficient matrix of the following system of equations

$$\begin{aligned} 2x - 3y + 4z &= -19 \\ 3x + 2y - z &= 4 \\ x + 5y + 4z &= -23 \end{aligned}$$

(a) find adj(A)
(b) Use adj (A) to find A^{-1} , if exists,
(c) Use A^{-1} to solve the given system.

6. Solve the system of equations by finding A^{-1} by method of cofactors:

(ii)
$$\begin{aligned} x + 2y + 8z &= 5 \\ -x - y \qquad &= 1 \\ x + 2y + 7z &= 4 \end{aligned}$$

3.8 Cramer's Rule

If A is nxn marix with $\det(A) \neq 0$, then the linear system $AX = B$ has a unique solution $X = (x_j)$ given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad ,j = 1,2,\ldots,n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

NOTE: If A is 3x3 matrix , then the solution of the system $AX = B$ is

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}, \quad z = \frac{\det(A_{31})}{\det(A)}$$

Example: Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution: $A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$\det(A) = -132$, $\det(A_1) = -36$, $\det(A_2) = -24$, $\det(A_3) = 12$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$

$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

NOTE: when $\det(A) = 0$, then there does not exist any solution of the system.

EXERCISE 6

Cramer's Rule

In problem 1 to 9 , use Cramer's Rule to solve the system of linear equations:

1. $$\begin{aligned} 2x + 2y &= 2 \\ 3x - y &= -6 \end{aligned}$$

2. $$\begin{aligned} 3x + 5y &= 21 \\ 2x + 3y &= 12 \end{aligned}$$

3. $$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 5y + 3z &= -2 \\ x + 8z &= 8 \end{aligned}$$ Ans: $x = 0, y = -1, z = 1$

4. $$\begin{aligned} 2x + 2y &= 1 \\ -2x + y + z &= 0 \\ 3x + z &= 1 \end{aligned}$$ Ans: $x = \frac{1}{4}, y = \frac{1}{4}, z = \frac{1}{4}$

5. $$\begin{aligned} 3x + 2y + z &= 7 \\ 2x - y + 3z &= 3 \\ 5x + 4y - 2z &= 1 \end{aligned}$$ Ans: $x = \frac{13}{7}, y = \frac{31}{7}, z = \frac{26}{7}$

6. $$\begin{aligned} 2x - 3y + 4z &= -19 \\ 3x + 2y - z &= 4 \\ x + 5y + 4z &= 23 \end{aligned}$$

7. $$\begin{aligned} 4y + 3z &= -2 \\ -2x + 5y - z &= 1 \\ 3x + 4y + 5z &= 6 \end{aligned}$$ Ans: $x = \frac{180}{41}, y = \frac{59}{41}, z = -\frac{106}{41}$

8. $$\begin{aligned} x + y + 2z &= 0 \\ -2x + 5y + 2z &= 1 \\ 3x + y + 4z &= -1 \end{aligned}$$ Ans: $x = -3, y = -2, z = \frac{5}{2}$

9. $$\begin{aligned} x - y + z + w &= -2 \\ -x + 3y + 2w &= 0 \\ y + z - w &= 1 \\ x + z - w &= 1 \end{aligned}$$