Ch 04-3 Numerical Integration I

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Outline

- Introduction to Numerical Integration
- The Trapezoidal Rule
- Simpson’s Rule
- Comparing the Trapezoidal Rule with Simpson’s Rule

Introduction to Numerical Integration

Numerical Quadrature

- The need often arises for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain.
- The basic method involved in approximating $\int_a^b f(x)\,dx$ is called numerical quadrature. It uses a sum $\sum_{i=0}^{n} a_i f(x_i)$ to approximate $\int_a^b f(x)\,dx$.

Introduction to Numerical Integration

Quadrature based on interpolation polynomials

- The methods of quadrature in this section are based on the interpolation polynomials.
- The basic idea is to select a set of distinct nodes $\{x_0, \ldots, x_n\}$ from the interval $[a, b]$.
- Then integrate the Lagrange interpolating polynomial

$$P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$$

and its truncation error term over $[a, b]$ to obtain:

Introduction to Numerical Integration

Quadrature based on interpolation polynomials (Cont'd)

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^b \sum_{i=0}^{n} f(x_i)L_i(x)\,dx + \int_a^b \prod_{i=0}^{n}(x-x_i)\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\,dx \\ &= \sum_{i=0}^{n} a_i f(x_i) + \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x-x_i) f^{(n+1)}(\xi(x))\,dx\end{aligned}$$

where $\xi(x)$ is in $[a,b]$ for each x and

$$a_i = \int_a^b L_i(x)\,dx, \quad \text{for each } i = 0, 1, \ldots, n$$

Introduction to Numerical Integration

Quadrature based on interpolation polynomials (Cont'd)

The quadrature formula is, therefore,

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

where

$$a_i = \int_a^b L_i(x)\,dx, \quad \text{for each } i = 0, 1, \ldots, n$$

and with error given by

$$E(f) = \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x))\,dx$$

Numerical Integration: Trapezoidal Rule

Derivation (1/3)

To derive the Trapezoidal rule for approximating $\int_a^b f(x)\,dx$, let $x_0 = a$, $x_1 = b$, $h = b - a$ and use the linear Lagrange polynomial:

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

Then

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)\right] dx \\ &\quad + \frac{1}{2}\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)\,dx.\end{aligned}$$

Numerical Integration: Trapezoidal Rule

Derivation (2/3)

The product $(x - x_0)(x - x_1)$ does not change sign on $[x_0, x_1]$, so the Weighted Mean Value Theorem for Integrals ▸ See Theorem can be applied to the error term to give, for some ξ in (x_0, x_1),

$$
\begin{aligned}
& \int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)\, dx \\
= \quad & f''(\xi) \int_{x_0}^{x_1} (x - x_0)(x - x_1)\, dx \\
= \quad & f''(\xi) \left[\frac{x^3}{3} - \frac{(x_1 + x_0)}{2} x^2 + x_0 x_1 x \right]_{x_0}^{x_1} \\
= \quad & -\frac{h^3}{6} f''(\xi)
\end{aligned}
$$

Numerical Integration: Trapezoidal Rule

Derivation (3/3)

Consequently, the last equation, namely

$$\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1)\, dx = -\frac{h^3}{6} f''(\xi)$$

implies that

$$\begin{aligned}\int_a^b f(x)\, dx &= \left[\frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1)\right]_{x_0}^{x_1} - \frac{h^3}{12} f''(\xi) \\ &= \frac{(x_1 - x_0)}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)\end{aligned}$$

Numerical Integration: Trapezoidal Rule

Using the notation $h = x_1 - x_0$ gives the following rule:

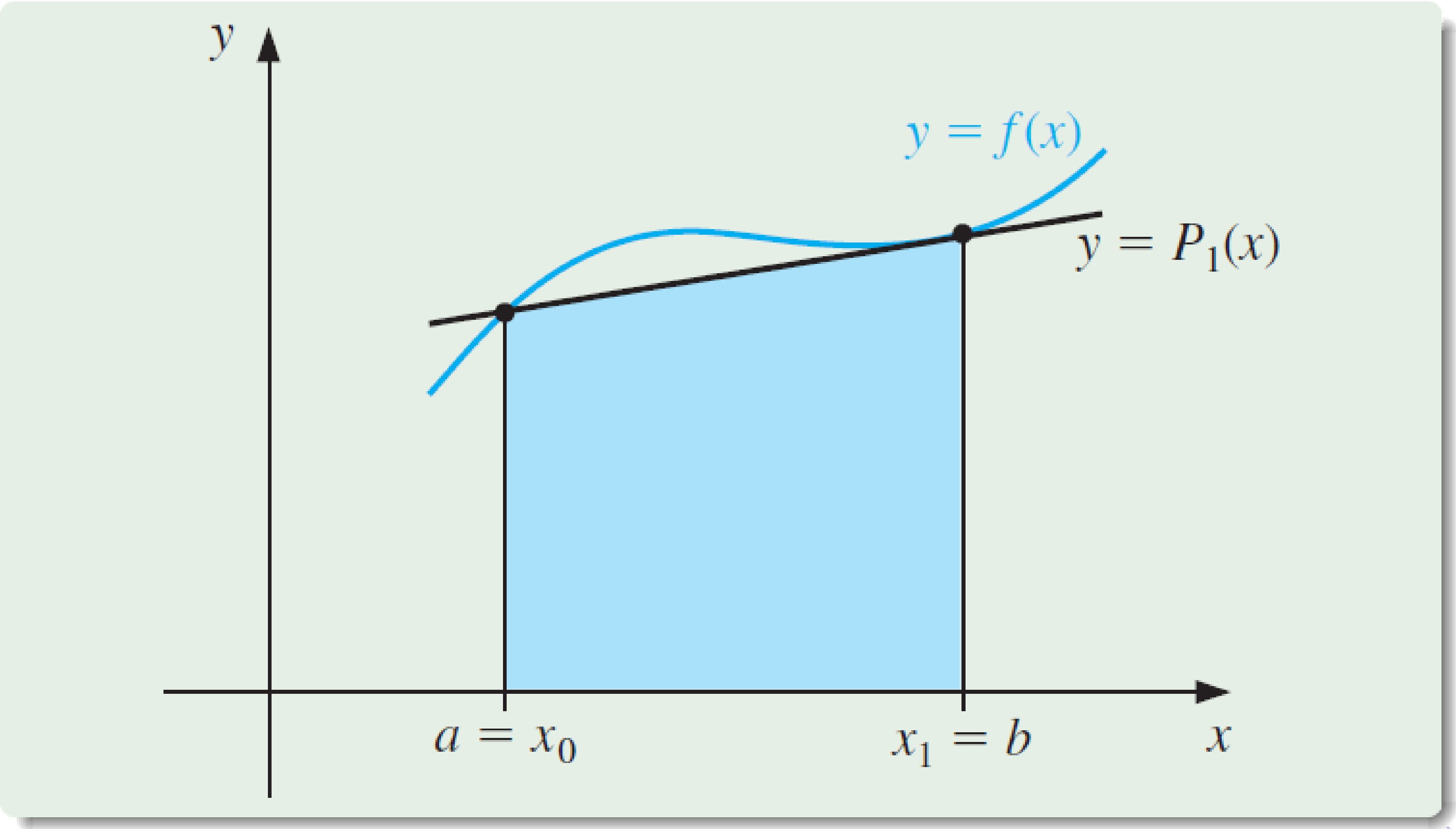
The Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12}f''(\xi)$$

Note:

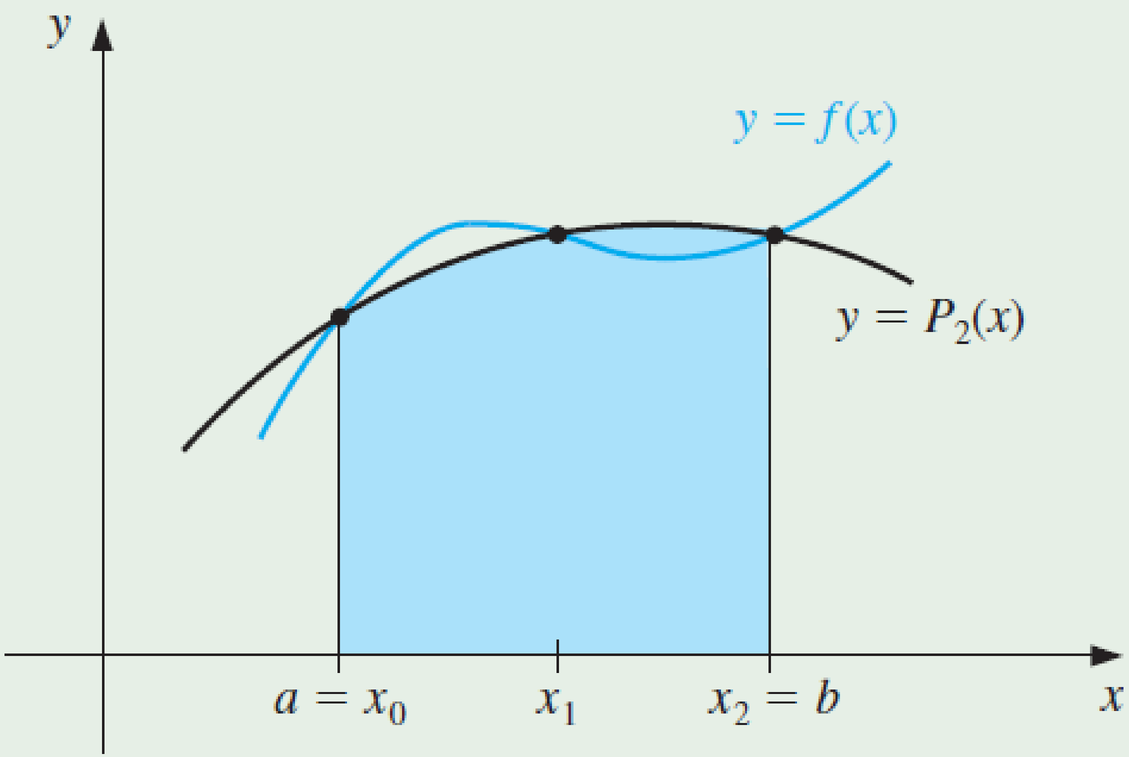
- The error term for the Trapezoidal rule involves f'', so the rule gives the exact result when applied to any function whose second derivative is identically zero, that is, any polynomial of degree one or less.
- The method is called the Trapezoidal rule because, when f is a function with positive values, $\int_a^b f(x)\,dx$ is approximated by the area in a trapezoid, as shown in the following diagram.

Trapezoidal Rule: The Area in a Trapezoid



Numerical Integration: Simpson's Rule

Simpson's rule results from integrating over $[a, b]$ the second Lagrange polynomial with equally-spaced nodes $x_0 = a$, $x_2 = b$, and $x_1 = a + h$, where $h = (b - a)/2$:



Numerical Integration: Simpson's Rule

Simpson's Rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}f^{(4)}(\xi)$$

The error term in Simpson's rule involves the fourth derivative of f, so it gives exact results when applied to any polynomial of degree three or less.

Trapezoidal Rule .v. Simpson's Rule

Example

Compare the Trapezoidal rule and Simpson's rule approximations to $\int_0^2 f(x)\,dx$ when $f(x)$ is

(a) x^2 (b) x^4 (c) $(x+1)^{-1}$

(d) $\sqrt{1+x^2}$ (e) $\sin x$ (f) e^x

Trapezoidal Rule .v. Simpson's Rule

Solution (1/3)

On $[0, 2]$, the Trapezoidal and Simpson's rule have the forms

$$\text{Trapezoidal:} \quad \int_0^2 f(x)\,dx \approx f(0) + f(2)$$

$$\text{Simpson's:} \quad \int_0^2 f(x)\,dx \approx \frac{1}{3}[f(0) + 4f(1) + f(2)]$$

Trapezoidal Rule .v. Simpson's Rule

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When $f(x) = x^2$ they give

$$\text{Trapezoidal:} \quad \int_0^2 f(x)\,dx \approx 0^2 + 2^2 = 4$$

$$\text{Simpson's:} \quad \int_0^2 f(x)\,dx \approx \frac{1}{3}[(0^2) + 4 \cdot 1^2 + 2^2] = \frac{8}{3}$$

Trapezoidal Rule .v. Simpson's Rule

Solution (2/3)

- The approximation from Simpson's rule is exact because its truncation error involves $f^{(4)}$, which is identically 0 when $f(x) = x^2$.
- The results to three places for the functions are summarized in the following table.

Trapezoidal Rule .v. Simpson's Rule

Solution (3/3): Summary Results

	(a)	(b)	(c)	(d)	(e)	(f)
$f(x)$	x^2	x^4	$(x+1)^{-1}$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's	2.667	6.667	1.111	2.964	1.425	6.421

Notice that, in each instance, Simpson's Rule is significantly superior.