Anatomy of a Cobb-Douglas Type Production/Utility Function in Three Dimensions

(A Visual Guide for Econ Majors)

Peter Fuleky
Department of Economics, University of Washington

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Decreasing returns to scale (Strongly concave $y/U$)

production/utility function = $a x_1^a x_2^b$
$a=1, \alpha=\beta=0.25$ - decreasing returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

diminishing marginal product/utility

increasing both inputs/goods by the same factor $(x_1^3, x_2^2)$

slice along a ray through the origin

decreasing returns to scale
Constant returns to scale (Weakly concave y/U)

production/utility function = \( a x_1^\alpha x_2^\beta \)
\( n=1, \alpha=\beta=0.5 \) - constant returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

diminishing marginal product/utility

increasing both inputs/goods by the same factor \( [x_1^3, x_2^2] \)

slice along a ray through the origin

constant returns to scale

Author: Peter Fuleky, University of Washington
Increasing returns to scale with diminishing marginal product/utility (Quasiconcave $y/U$)

Production/utility function: $a x_1^\alpha x_2^\beta$

$\alpha = 1, \beta = 0.75$ - increasing returns to scale

Holding output/utility fixed

Convex isoquants/indifference curves

Holding one input/good fixed

Slice parallel to axis

Diminishing marginal product/utility

Increasing both inputs/goods by the same factor: $x_1^3, x_2^2$

Slice along a ray through the origin

Increasing returns to scale
Increasing returns to scale with linear marginal product/utility (Quasiconcave $y/U$)

production/utility function $- \alpha x_1^\alpha x_2^\beta$
\(\alpha=1, \beta=1\) - Increasing returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

constant marginal product/utility

 Increasing both inputs/goods by the same factor $[x_1^3, x_2^2]$}

slice along a ray through the origin

increasing returns to scale
Increasing returns to scale with increasing marginal product/utility (Quasiconcave $y/U$)
Non-symmetric production/utility function with constant returns to scale

production/utility function: $a x_1^\alpha x_2^\beta$

$a = 1$, $\alpha = 0.75$, $\beta = 0.25$

isquants/indifference curves

marginal properties (intensive margin)

meshgrid

tscale properties (extensive margin)
Non-symmetric production/utility function with increasing returns to scale
Profit maximization: production function with decreasing returns to scale

production function: \( y = x_1^a x_2^b \)
\( a = 1, \quad b = 0.25 \) - decreasing returns to scale

cost function: \( C = \frac{w_1 x_1 + w_2 x_2}{w_1 - w_2 - 1} \)

the two functions combined

the vertical difference: value of output - cost = profit
only profit > 0 displayed

slice along a ray

with decreasing returns to scale, an interior solution for maximum profit can be achieved
Profit maximization: production function with constant returns to scale

production function: \( y = a x_1^\alpha x_2^\beta \)
\( a=1, \alpha=\beta=0.5 \) - constant returns to scale

cost function: \( C = w_1 x_1 + w_2 x_2 \)
\( w_1=2, w_2=3 \)

In a competitive market, price and wage adjust:
\( p = 2 \sqrt{w_1 w_2} = 2 \sqrt{6} \)
profit = value of output - cost = zero

In a non-competitive market, we don't have to impose zero profit

profit in a non-competitive market
profit = value of output - cost

In a non-competitive market, profit could be infinite (but with increasing production, decreasing returns to scale eventually kick in)
Utility maximization: utility function with decreasing returns to scale

utility function: \( U = a x_1^\alpha x_2^\beta \)
\( \alpha = 1, \beta = 0.25 \) - decreasing returns to scale

expenditure held constant at: \( m^0 = p_1 x_1 + p_2 x_2 \)
\( p_1 = 2, p_2 = 3 \)

maximize utility subject to expenditure constraint

constrained maximum utility: the tangency point of the constraint and the indifference curve with the maximum value
Utility maximization: utility function with increasing returns to scale

utility function: $U = \alpha x_1^a x_2^\beta$

$a=1, \alpha=\beta=1.25$ - increasing returns to scale

expenditure held constant: $M^e = p_1 x_1 + p_2 x_2$
$p_1=2, p_2=3$

maximize utility subject to expenditure constraint

constrained maximum utility:
the tangency point of the constraint and the indifference curve with the maximum value
Expenditure minimization: utility function with constant returns to scale

Expenditure function: \( M = p_1 x_1 + p_2 x_2 \)
- \( p_1 = 2 \), \( p_2 = 3 \)

Utility function: \( U^0 = \alpha x_1^\alpha x_2^\beta \)
- \( \alpha = 1 \), \( \beta = 0.5 \) - constant returns to scale

Minimize expenditure subject to utility constraint

Constrained minimum expenditure: the tangency point of the constraint and the isocost curve with the minimum value
Literature and further reading:

The Structure of Economics, 3rd ed., Eugene Silberberg
MATLAB Documentation, MathWorks