ON THE ENDOMORPHISM RING OF A CHAIN MODULE

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Abstract

The aim of this paper is to further develop the theory of the well-known Schur's lemma on endomorphism rings for application to wider structures. We show that for a commutative Noetherian chain ring R and Noetherian chain modules M and N, the left R-module $Hom_R(M, N)$ will be a chain module. Then, we conclude that the endomorphism ring $End_R(M)$ of the Noetherian chain module M over such R is a Noetherian chain ring. As a special case, we consider that the dual R- module $M^* = Hom_R(M, R)$ of the Noetherian chain R-module M is a Noetherian chain module.

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0 Introduction

The chain ring is a generalization of the concepts of the division ring and commutative valuation rings. Several approaches, both algebraic and geometric, have been employed in studying this concept (see [2]). This research will

be supported by Alolaiyan's thesis " *The structure of chain rings*" [1], which is focused on chain rings, generalized chain rings and their relationship with Galois groups.

In this paper, we shall recall a parallel concept for a module over a chain ring. A chain module indicates a module with a unique series of submodules. In other words, even if two submodules of a module M are arbitrary, they should be comparable. The main objective is to investigate the structure of the endomorphism ring of a Noetherian chain module. This investigation is followed by a study of the behavior of a chain ring under the assumptions that the module under construction is a Noetherian chain module and that the coefficients come from a commutative Noetherian chain ring.

This paper is organized as follows. In Section 1, we discuss the set of ideals in a commutative Noetherian chain ring and the set of submodules in the chain modules. Then, in Section 2, we present the main results for the endomorphism ring of a Noetherian chain module and the dual of a chain module.

We assume that all rings are associative commutative with identity and that all modules are unitary modules. By an R-module (R-submodule), we refer to the left R-module (R-submodule), where R is a ring with identity and is a commutative Noetherian chain ring. For more information about chain rings, the reader can consult the survey article [2].

1 Preliminaries and Definitions

In this section, we recast some definitions and standard facts and describe certain properties that characterize the set of ideals and submodules in specific algebraic structures, which are then used to reach our conclusion.

Definition 1.1 *A ring R is called a* **left chain ring** *if for all $a, b \in R$, we have either $Ra \subseteq Rb$ or $Rb \subseteq Ra$. Similarly, R is called a* **right chain ring** *if for all $a, b \in R$, we have either $aR \subseteq bR$ or $bR \subseteq aR$. The ring R is called a* **chain ring** *if its ideals are comparable.*

Definition 1.2 *A left (right) module M over a ring R is called a* ***left (right) chain module*** *if any two left (right) submodules of M are comparable.*

Definition 1.3 *Let R be a commutative ring, and let M, N be R-modules. Then, the module of all module homomorphisms from M to N is called the* **homomorphism R-module** *from M to N and is denoted by $Hom_R(M, N)$.*

Definition 1.4 *Let R be a commutative ring, and let M be an R-module. Then, the ring of all module homomorphisms from M to M is called the* **endomorphism ring** *of M and is denoted by $End_R(M)$.*

Definition 1.5 *Let R be a commutative ring, and let M be an $R-$module. Then, the module of all module homomorphisms from M to R is called the* **dual module** *of M and is denoted by $M^* = Hom_R(M, R)$.*

Proposition 1.6 *Let R be a commutative Noetherian chain ring. Then, its ideals form a unique series of the form $R \supset \langle\theta\rangle \supset \langle\theta^2\rangle \supset \cdots$, for some $\theta \in R$.*

Proof: We will show that any Noetherian chain ring is a principle ideal ring. As R is Noetherian, all of its ideals are finitely generated. Now, let L be any ideal of R, and suppose that a, b generates L. Because R is a chain ring, either$< a >\subset< b >$ or $< b >\subset< a >$, and hence, L is principle ideal. We can easily use induction to show that any finitely generated ideal is principle. As R is a chain ring, it is also a local ring, and hence, $J(R)$ is the maximal ideal of R. Based on the above, $J(R)$ is generated by a single element, which we shall call θ. Thus, it is clear that

$$R \supset \langle\theta\rangle \supset \langle\theta^2\rangle \supset \cdots.$$

To show that the above result is a series, suppose that there is an ideal K in R such that

$$J^i(R) =< \theta^i >\supset K \supset< \theta^{i+1} >= J^{i+1}(R),$$

for some i. Because R is a chain ring, K is principle. Let $K =< a >$; then, $a \in J^i(R)$, $a = r\theta^i$, r is a unit; otherwise, if r in $J(R)$, then $a \in J(R)^{i+1}$, which is a contradiction. Thus, $r^{-1}a = \theta^i$ and $\theta \in K$, which implies that the above series is a unique series.

In the following proposition, the Jacobson radical of an R-module M denotes the intersection of all maximal R-submodules of M.

Proposition 1.7 *Let R be a ring and M be a chain R-module. Then, the Jacobson radical $Rad(M)$ of M is a maximal R-submodule.*

Proposition 1.8 *Let R be a ring and M be a Noetherian chain $R-$module. Then, the Jacobson radical $Rad(M)$ of M is generated by a single element.*

Proof: Suppose that $Rad(M)$ cannot be generated by one element and is thus generated by at least two elements, say, u, v. Therefore, Ru is not contained in Rv, and Rv is not contained in Ru, which produces a contradiction, as M is a chain R-module. Thus, $Rad(M)$ is generated by a single element.

Proposition 1.9 *Let R be a commutative Noetherian chain ring, and let M be a Noetherian chain R-module. Then, the set of all R-submodules of M forms a unique series as follows:*

$$M =< w >\supset \langle \theta w \rangle \supset \langle \theta^2 w \rangle \supset \cdots ,$$

where $J(R)w = Rad(M)$*, and* $J(R) =< \theta >$.

Proof: Let N be an R-submodule of M . It is clear that N is generated by a single element. Let $N = Ra$ for some a in M. As $N \subset RadM$, $a = r\theta w$, for some $r \in R$, $r = s\theta^i, s$ is a unit; therefore, $N \subseteq J(R)^{i+1}w$. Moreover, $\theta^{i+1}w = s^{-1}r\theta w = s^{-1}a \in N$, and hence, $N = J(R)^{i+1}w$. Thus, all R-submodules of M form a chain as follows: $M =< w >\supset \langle \theta w \rangle \supset \langle \theta^2 w \rangle \supset \cdots$, where $J(R)w = Rad(M)$ and $J(R) =< \theta >$.

2 Main Results

It is well known that for a ring R and a simple $R-module$ M, $End_R(M)$ is a division ring. We shall replace the simplicity property with the chain property for an R-module M, assume that R itself is a chain ring, and study the ring $End_R(M)$. In this section, we would like to investigate certain properties of the R-module $Hom_R(M, N)$, where M, N are Noetherian chain R-modules. Then, we will discuss the endomorphism ring of M over a Noetherian chain ring R and show that the endomorphism ring $End_R(M)$ is a chain ring. Moreover, we shall study the special case in which $Hom_R(M, N)$ is the dual R-module $M^* = Hom_R(M, R)$.

In the following lemma, we will attempt to find a characterization for the left submodules of an $R - module$ $Hom_R(M, N)$, where R is a commutative Noetherian chain ring and M, N are Noetherian chain R-modules.

Lemma 2.1 *Let R be a commutative Noetherian chain ring, and let M, N be Noetherian chain R-modules. Then, each left submodule L of an R-module $K = Hom_R(M, N)$ is of the form $L = Rf$, for some $f \in K$.*

Proof: Let L be a left submodule of an R-module $K = Hom_R(M, N)$; thus, any $g \in L$ takes any left submodule of M to a left submodule in N. The left submodules of M, N form a unique series as follows:

$$M =< w >\supset \langle \theta w \rangle \supset \langle \theta^2 w \rangle \supset \cdots,$$

where $J(R)w = Rad(M)$, and $J(R) =< \theta >$,

$$N =< v >\supset \langle \theta v \rangle \supset \langle \theta^2 v \rangle \supset \cdots,$$

where $J(R)v = Rad(N)$, and $J(R) =< \theta >$. As g is an $R - module$ homomorphism, $g(w) = r\theta^i v, r$ is a unit in R, $i \in 0, 1, 2, \cdots$. Let f be the element in L for which $f(w) = r\theta^i v, r$ is a unit in R and i is the smallest power

for all elements in L. If h in L has the above property, such that $f(w) = r\theta^i v, r$ is a unit in R and $h(w) = s\theta^i v, s$ is a unit in R, then $rs^{-1}h = f$. Thus, we can take $r = 1, f(w) = \theta^i v$, where i is the smallest integer for all elements in L; thus, $Rf \subseteq L$. Let g be any element in L. Then, $g(w) = t\theta^j v, t$ is a unit in R, $j \geq i$, such that $t\theta^{j-i} f = g, g \in Rf$. Hence, $L = Rf$.

We will call the homomorphism f in the above proof the ***minimal element*** in L and i the ***degree of*** f. Now, we will study the ideals in the ring $End_R(M)$, where R is a Noetherian chain ring and M is a Noetherian chain R-module.

Lemma 2.2 *Let R be a commutative Noetherian chain ring, and let M be a Noetherian chain R-module. Then, $End_R(M)$ is a principle ideal ring.*

Proof: Let I be an ideal of the ring $End_R(M)$, so that any $f \in I$ takes any submodule of M to a submodule of M. The submodules of M form the following unique series:

$$M =< w >\supset \langle \theta w\rangle \supset \langle \theta^2 w\rangle \supset \cdots ,$$

where $J(R)w = Rad(M)$ and $J(R) =< \theta >$.

Therefore, because f is an $R-module$ endomorphism, $f(w) = r\theta^i w, r$ is a unit in R, $i \in 0, 1, 2, \cdots$. Let g be the minimal element in I such that $g(w) = \theta^i w$. Then, for any $h \in End_R(M)$, we have $h(w) = t\theta^j w$, and thus,

$$hg(w) = h(\theta^i w) = \theta^i h(w) = \theta^i t\theta^j w = t'\theta^{i+j} w,$$

for some $t' \in R$ and hence $hg \in I$. Similarly, we can show that $gh \in I$ and $< g >\subseteq I$. Now, let f be any element in I. Then, $f(w) = t\theta^j w, t$ is a unit in R, $j \geq i$, which implies that $t^{-1}\theta^{j-i} g = f$, and thus $f \in< g >$. Hence, $I =< g >$, and $End_R(M)$ is a principle ideal ring.

Theorem 2.3 *If R is a commutative Noetherian chain ring, and M, N are Noetherian chain R-modules, then the module $Hom_R(M, N)$ is a Noetherian chain R-module. Moreover, if N is of finite length, then the length of the composition series of $Hom_R(M, N)$ is equal to the length of the composition series of N.*

Proof: Suppose that the composition series of M and N are as follows:

$$M =< w >\supset \langle \theta w\rangle \supset \langle \theta^2 w\rangle \supset \cdots,$$

$$N =< v >\supset \langle \theta v\rangle \supset \langle \theta^2 v\rangle \supset \cdots,$$

where $J(R) =< \theta >$. Let I be a submodule of $Hom_R(M, N)$, and let $f \in I$. Then, the image of any submodule of M under f is a submodule of N. Assume that f is the minimal element in I, so that $f(w) = \theta^i v$, $i \in 0, 1, 2, \cdots$ is the smallest power for all elements in I. Let K be any submodule in $Hom_R(M, N)$. Then, by lemma 2.1, $K = Rg, g$ is the minimal element in K. Now, $g(w) = \theta^j v$, $j \in 0, 1, 2, \cdots$. If $i \leq j$, $g(w) = \theta^j v = \theta^{j-i}\theta^i v = \theta^{j-i} f(w)$, and hence $Rg \subseteq Rf$. Conversely, if $j \leq i$, then $Rf \subseteq Rg$. Thus, $Hom_R(M, N)$ is a Noetherian chain R-module. Now, we will show that the unique series of $Hom_R(M, N)$ is as follows:

$$Hom_R(M, N) = Rh \supset R\theta h \supset R\theta^2 h \subset R\theta^3 h \subset \cdots,$$

where $h(w) = v$. Let K be any submodule of $Hom_R(M, N)$. Then, $K = Rg$, $g(w) = \theta^j v$, $j \in 0, 1, 2, \cdots$, is the minimal element in K, so $g = \theta^j h$, and hence, the above series is a unique series.

It is clear that if N is of finite length, then the length of the composition series of $Hom_R(M, N)$ is equal to the length of the composition series of N.

Here, we present a development of Schur's lemma, asserting that for a ring R and a simple $R - module$ M, $End_R(M)$ is a division ring.

Corollary 2.4 *Let R be a commutative Noetherian chain ring, and let M be a Noetherian chain R-module. Then, the endomorphism ring $End_R(M)$ is a Noetherian chain ring.*

Proof: Let I, L be two ideals of $End_R(M)$. Then $I =< g >, L =< h >$, where g, h are the minimal elements in I and L and i, j are the degrees of g, h respectively. Assume that $i \leq j$. Then, $g(w) = \theta^i w, h(w) = \theta^j w = \theta^{i+(j-i)} w = \theta^{j-i} g(w)$, and hence, $h = \theta^{j-i} g \in< g >, L \subseteq I$. Because I, L are arbitrary, we conclude that $End_R(M)$ is a chain ring.

Corollary 2.5 *Let R be a commutative Noetherian chain ring, and let M be a Noetherian chain $R - module$. Then, $Hom_R(M, N)$ is an Artinian R-module if R is an Artinian ring.*

Proof: From the proof for Theorem 2.3, we can find that the length of the composition series of R as a ring is greater than or equal to the length of composition series of the R-module $Hom_R(M, N)$. Hence, if R is an Artinian ring, then $Hom_R(M, N)$ is an Artinian $R-$module.

In general, the converse of the above corollary is not true. For example, if we assume that N has a finite length as an R-module, say n, and R is a chain ring that is not Artinian, then the length of the composition series of $Hom_R(M, N)$ is equal to n, and hence $Hom_R(M, N)$ is an Artinian $R-module$.

Corollary 2.6 *Let R be a commutative Noetherian chain ring and M a Noetherian chain R-module. Then, $End_R(M)$ is an Artinian ring if R is an Artinian ring.*

To obtain the following corollary, we need to assert that if R is a Noetherian chain ring, then R is a Noetherian chain R-module. Assuming that R is a chain ring, we aim to show that R is itself a chain R-module. Suppose that $a, b \in R$. As $J(R) =< \theta >$ is the maximal left ideal of R, $Ra, Rb \subset J(R)$, $a = r\theta^i$ for some $i \in 0, 1, 2, \cdots$, r is a unit with the smallest power of θ in Ra. Similarly, for Rb, we have $b = t\theta^j$, $j \in 0, 1, 2, \cdots$, t is a unit with the smallest power of θ in Rb. Now (without loss of generality), assume that $i \leq j$. Then, $b = t\theta^j = t\theta^{j-i}r^{-1}r\theta^i = t\theta^{j-i}r^{-1}a \in Ra$, which implies that $Rb \subseteq Ra$. Hence, R is a Noetherian chain R-module.

Now, for the dual R-module M^* of M, we will set $N = R$ and obtain the following result.

Corollary 2.7 *Let R be a commutative Noetherian chain ring, and let M be a Noetherian chain R-module. Then, $M^* = Hom_R(M, R)$ is a Noetherian chain module. If R is of finite length as an R-module, then the length of the composition series of M^* is equal to the length of the composition series of the left ideals in R.*

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