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REGULAR ARTICLE

One- and two-sample Bayesian prediction intervals based on progressively Type-II censored data

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Abstract In this article, one- and two-sample Bayesian prediction intervals based on progressively Type-II censored data are derived. For the illustration of the developed results, the exponential, Pareto, Weibull and Burr Type-XII models are used as examples. Some of the previous results in the literature such as Dunsmore (Technometrics 16:455–460, 1974), Nigm and Hamdy (Commun Stat Theory Methods 16:1761–1772, 1987), Nigm (Commun Stat Theory Methods 18:897–911, 1989), Al-Hussaini and Jaheen (Commun Stat Theory Methods 24:1829–1842, 1995), Al-Hussaini (J Stat Plan Inference 79:79–91, 1999), Ali Mousa (J Stat Comput Simul 71: 163–181, 2001) and Ali Mousa and Jaheen (Stat Pap 43:587–593, 2002) can be achieved as special cases of our results. Finally, some numerical computations are presented for illustrating all the proposed inferential procedures.

Keywords Order statistics · Progressively Type-II right censored sample · Bayesian prediction · Exponential (θ) model · Pareto (α, β) model · Weibull (α, β) model · Burr Type-XII (α, β) model

1 Introduction

In many practical problems, one would wish to use previous data to predict a future observation from the same population. One way to do this is to construct an interval which will contain the future observation with a specified probability. This interval

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is called a prediction interval. Bayesian prediction bounds for future observations have been discussed by several authors, including Dunsmore (1974); Aitchison and Dunsmore (1975); Nigm and Hamdy (1987); Nigm (1988, 1989); Patel (1989); Geisser (1993); AL-Hussaini and Jaheen (1995); AL-Hussaini (1999); Ali Mousa (2001); Ali Mousa and Jaheen (2002); Jaheen (2003); AL-Hussaini and Ahmed (2003); Raqab and Madi (2005); Wu et al. (2006); Abdel-Aty et al. (2007); Amin (2008); Kundu (2008); AL-Hussaini and Al-Awadhi (2010).

In reliability analysis, experiments often terminate before all units on test have failed due to cost and time considerations. In such cases, failure information is available only on part of the sample, and on all units that had not failed, one has only partial information. Such data are said to be censored data. There are several forms of censored data. One of the most common form of censoring is Type-II censoring. In Type-II censoring, a total of n units are placed on test, but instead of continuing until all n units have failed, the test is terminated at the time of the mth ($1 \leq m \leq n$) failure. A generalization of Type-II censoring is progressive Type-II censoring. Under this scheme, n units are placed on test and only m complete failures are going to be observed. When the first failure is observed, R_1 of the surviving units are randomly selected and removed. At the second observed failure, R_2 of the surviving units are randomly selected and removed. The experiment finally terminates at the time when the mth failure is observed and all remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ surviving units are removed. If $R_1 = R_2 = \cdots = R_{m-1} = 0$, then $R_m = n - m$, which corresponds to Type-II right censoring. If $R_1 = R_2 = \cdots = R_{m-1} = R_m = 0$, then $m = n$, which corresponds to the complete sample (ordinary order statistics).

Let $\underline{\mathbf{X}} = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ denote a progressively Type-II censored sample from a continuous distribution with distribution function $F(x) \equiv F(x|\theta)$ and density function $f(x) \equiv f(x|\theta)$, where the parameter $\theta \in \Theta$ may be a real-vector, with censoring scheme $\underline{\mathbf{R}} = (R_1, R_2, \ldots, R_m)$. Then the joint density function of all m progressively Type-II censored order statistics is given by

$$f_{X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}}(\underline{\mathbf{x}}) = K \prod_{i=1}^{m} f(x_i)[\bar{F}(x_i|\theta)]^{R_i}, \tag{1.1}$$

where $\bar{F}(x|\theta) = 1 - F(x|\theta)$ is the survival function (SF), $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_m)$ and $K = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - R_3 - \cdots - R_{m-1} - m + 1)$.

Several authors have studied progressive Type-II censoring and properties of order statistics arising from such a progressively censored life-test. Some key references are Balakrishnan and Cohen (1991); Viveros and Balakrishnan (1994); Balakrishnan and Sandhu (1995); Aggarwala and Balakrishnan (1998); Balakrishnan and Aggarwala (2000); Basak et al. (2006); Balakrishnan (2007); Raqab (2010).

In this article, we consider a general form for the underlying distribution and a general conjugate prior, and describe a general procedure for determining the Bayesian prediction intervals for future life-lengths based on observed progressively Type-II censored data. In Sect. 2, we use the general exponential form for the underlying distribution and the general exponential prior family to derive the posterior density function of θ, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$. In Sect. 3, we derive the one-sample

Bayesian prediction intervals for all censored units in all m stages of censoring from a progressively Type-II censored data, and we also derive the two-sample Bayesian prediction intervals when both the (observed) informative sample and the (unobserved) future sample are progressively Type-II censored samples from the same distribution. In Sect. 4, we present the results for exponential(θ), Pareto(α, β), Weibull(α, β) and Burr Type-XII(α, β) models as illustrative examples. Finally, in Sect. 5, we present numerical results.

2 Prior and posterior distributions

Since the SF $\bar{F}(x|\theta) = 1 - F(x|\theta)$ corresponding to any cumulative distribution function (CDF) $F(x|\theta)$, $\theta \in \Theta$, can be written in the form

$$\bar{F}(x|\theta) = \exp[-\lambda(x;\theta)], \tag{2.1}$$

where $\lambda(x;\theta) = -\ln \bar{F}(x|\theta)$, we shall consider the underling population SF to be given by (2.1). Of course, some conditions need to be imposed so that $\bar{F}(x|\theta)$ is a valid SF. These conditions are: $\lambda(x;\theta)$ is continuous, monotone increasing and differentiable function, with $\lambda(x;\theta) \to 0$ as $x \to -\infty$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. The probability density function (pdf) corresponding to (2.1) is given by

$$f(x|\theta) = \lambda'(x;\theta)\exp[-\lambda(x;\theta)], \tag{2.2}$$

where $\lambda'(x;\theta)$ is the first derivative of $\lambda(x;\theta)$ with respect to x.

With an appropriate choice of $\lambda(x;\theta)$ (notice that the derivative of $\lambda(x;\theta)$ with respect to x is the hazard rate function), several distributions that are used in reliability studies can be obtained as special cases such as exponential(θ), Pareto(α, β), Weibull(α, β) and Burr Type-XII(α, β) distributions. Appropriate conditions need to be imposed on $\lambda(x;\theta)$ to suit the domain on which $\bar{F}(x|\theta)$ in (2.1) is defined. For example, if $\bar{F}(x|\theta)$ is defined only on the positive half of the real line (as for an exponential, Weibull and Burr Type-XII distributions), then $\lambda(x;\theta) \to 0$ as $x \to 0^+$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. If $\bar{F}(x|\theta)$ is defined on (β, ∞) (as in Pareto distribution), then $\lambda(x;\theta) \to 0$ as $x \to \beta^+$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. The exponential form of the SF in (2.1) provides some flexibility in developing general results as done in the following sections.

Upon using (2.1) and (2.2) in (1.1), we obtain the likelihood function as

$$L(\theta;\underline{\mathbf{x}}) = K\prod_{i=1}^{m}\lambda'(x_i,\theta)\exp\left[-\sum_{j=1}^{m}(R_j+1)\lambda(x_j;\theta)\right]. \tag{2.3}$$

For the Bayesian prediction setup, we need a suitable prior parameter distribution. We consider here a general conjugate prior, suggested by AL-Hussaini (1999), that is given by

$$\pi(\theta;\delta) \propto C(\theta;\delta)\exp[-D(\theta;\delta)], \tag{2.4}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution under consideration and δ is the vector of prior parameters. The general exponential prior family in (2.4) includes several priors used in the literature as special cases.

Then, from (2.3) and (2.4), the posterior density function of θ, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, becomes

$$\pi^*(\theta|\underline{\mathbf{x}}) = I^{-1}\eta(\theta; \underline{\mathbf{x}}) \exp[-\zeta(\theta; \underline{\mathbf{x}})] \tag{2.5}$$

where

$$\eta(\theta; \underline{\mathbf{x}}) = C(\theta; \delta)\prod_{i=1}^{m}\lambda'(x_i, \theta), \tag{2.6}$$

$$\zeta(\theta; \underline{\mathbf{x}}) = \sum_{i=1}^{m}(R_i + 1)\lambda(x_i; \theta) + D(\theta; \delta), \tag{2.7}$$

and

$$I = \int_{\theta\in\Theta} \eta(\theta; \underline{\mathbf{x}}) \exp[-\zeta(\theta; \underline{\mathbf{x}})]d\theta. \tag{2.8}$$

3 Bayesian prediction

3.1 One-sample Bayesian prediction

Let $X_{j:R_k}$ denote the j-th order statistic out of R_k removed units at stage k, for $j = 1, 2, \ldots, R_k$ and $k = 1, 2, \ldots, m$. Then, the conditional density function of $X_{j:R_k}$, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, is given by Basak et al. (2006) as

$$f(x|\underline{\mathbf{x}}) = f(x|x_k) = \frac{R_k!}{(j-1)!(R_k-j)!}\frac{(F(x)-F(x_k))^{j-1}(1-F(x))^{R_k-j}f(x)}{(1-F(x_k))^{R_k}}, \tag{3.1}$$

where $x > x_k$, $j = 1, 2, \ldots, R_k$ and $k = 1, 2, \ldots, m$.

Theorem 3.1 *Suppose* $\underline{\mathbf{X}} = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ *is a progressively Type-II censored sample with progressive censoring scheme* $\underline{\mathbf{R}} = (R_1, R_2, \ldots, R_m)$. *Then, the* $100\tau\%$ *Bayesian prediction bounds for* $X_{j:R_k}$ *based on this progressively Type-II censored sample are obtained by solving the following equations with respect to* t:

$$\begin{aligned} &D_j(R_k)I^{-1}\sum_{w=0}^{j-1}\frac{c_w(j)}{(R_k - j + w + 1)}\int_{\theta\in\Theta}\eta(\theta; \underline{\mathbf{x}}) \\ &\quad\times\exp[-(\zeta(\theta; \underline{\mathbf{x}})+(R_k-j+w+1)(\lambda(t; \theta)-\lambda(x_k; \theta)))]d\theta = \begin{cases}(1+\tau)/2, \\ (1-\tau)/2,\end{cases} \end{aligned} \tag{3.2}$$

where $j = 1, 2, \ldots, R_k$, $k = 1, 2, \ldots, m$, $D_j(R_k) = j\binom{R_k}{j}$, $c_w(j) = (-1)^w\binom{j-1}{w}$, $\eta(\theta; \underline{\mathbf{x}})$, $\zeta(\theta; \underline{\mathbf{x}})$ *and* I *are given as* (2.6), (2.7) *and* (2.8), *respectively.*

Proof Upon using (2.1) and (2.2) in (3.1), we obtain the conditional density function of $X_{j:R_k}$, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, to be

$$\begin{aligned} f(x|\underline{\mathbf{x}}) &= \frac{R_k!}{(j-1)!(R_k-j)!}\lambda'(x;\theta)\exp[R_k\lambda(x_k;\theta)] \\ &\quad \times(\exp[-\lambda(x_k;\theta)] - \exp[-\lambda(x;\theta)])^{(j-1)}\exp[-(R_k-j+1)(\lambda(x;\theta))] \\ &= D_j(R_k)\sum_{w=0}^{j-1} c_w(j)\lambda'(x;\theta)\exp[-(R_k-j+w+1)(\lambda(x;\theta)-\lambda(x_k;\theta))]. \end{aligned} \tag{3.3}$$

By combining the posterior density in (2.5) with the conditional density in (3.3), and then integrating out the parameter θ, we obtain the Bayesian predictive density function of $X_{j:R_k}$, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, to be

$$\begin{aligned} p(x|\underline{\mathbf{x}}) &= \int\limits_{\theta\in\Theta} f(x|\underline{\mathbf{x}})\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= D_j(R_k)I^{-1}\sum_{w=0}^{j-1} c_w(j)\int\limits_{\theta\in\Theta}\lambda'(x;\theta)\eta(\theta;\underline{\mathbf{x}}) \\ &\quad \times\exp[-(\zeta(\theta;\underline{\mathbf{x}}) + (R_k-j+w+1)(\lambda(x;\theta)-\lambda(x_k;\theta)))]d\theta. \end{aligned} \tag{3.4}$$

From which, we simply obtain the predictive SF of $X_{j:R_k}$, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, as

$$\begin{aligned} P(X_{j:R_k} > t|\underline{\mathbf{x}}) &= \int_t^\infty p(x|\underline{\mathbf{x}})dx \\ &= D_j(R_k)I^{-1}\sum_{w=0}^{j-1}\frac{c_w(j)}{(R_k-j+w+1)}\int\limits_{\theta\in\Theta}\eta(\theta;\underline{\mathbf{x}}) \\ &\quad \times\exp[-(\zeta(\theta;\underline{\mathbf{x}}) + (R_k-j+w+1)(\lambda(t;\theta)-\lambda(x_k;\theta)))]d\theta. \end{aligned} \tag{3.5}$$

Consequently, the $100\tau\%$ Bayesian prediction bounds for $X_{j:R_k}$ based on the progressively Type-II censored sample are obtained by solving the following equations with respect to t:

$$P(X_{j:R_k} > t|\underline{\mathbf{x}}) = \begin{cases} (1+\tau)/2, \\ (1-\tau)/2, \end{cases}$$

where $P(X_{j:R_k} > t|\underline{\mathbf{x}})$ is as in (3.5). □

Corollary 3.2 *In the case when the observed sample is a Type-II right censored (i.e., $R_i = 0$, $1 \le i \le m-1$ and $R_m = n-m$), then the predictive SF of $X_{m+j:n}$ reduces to expression* (17) *of* AL-Hussaini (1999).

3.2 Two-sample Bayesian prediction

Let $\underline{\mathbf{Y}} = (Y_{1:\ell:N}, Y_{2:\ell:N}, \ldots, Y_{\ell:\ell:N})$ be a future independent progressively Type-II censored sample from the same population with censoring scheme $\underline{\mathbf{S}} = (S_1, S_2, \ldots, S_\ell)$. Then, the marginal density function of $Y_{s:\ell:N}$ is given by Balakrishnan et al. (2002) as

$$f_{Y_{s:\ell:N}}(y_s) = c(N,s)\sum_{q=0}^{s-1} c_{q,s-1}(1-F(y_s))^{T_{q,s}-1} f(y_s), \tag{3.6}$$

where $1 \le s \le \ell$,

$$c(N,s) = N(N-S_1-1)\cdots(N-S_1-\cdots-S_{s-1}-s+1),$$
$$c_{q,s-1} = \frac{(-1)^q}{\left[\prod_{u=1}^{q}\sum_{v=s-q}^{s-q+u-1}(S_v+1)\right]\left[\prod_{u=1}^{s-q-1}\sum_{v=u}^{s-q-1}(S_v+1)\right]},$$

and $T_{q,s} = N - S_1 - \cdots - S_{s-q-1} - s + q + 1$.

Theorem 3.3 *Suppose $\underline{\mathbf{X}} = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ is the informative progressively Type-II censored sample with progressive censoring scheme $\underline{\mathbf{R}} = (R_1, R_2, \ldots, R_m)$ and $\underline{\mathbf{Y}} = (Y_{1:\ell:N}, Y_{2:\ell:N}, \ldots, Y_{\ell:\ell:N})$ is a future independent progressively Type-II censored sample from the same population with censoring scheme $\underline{\mathbf{S}} = (S_1, S_2, \ldots, S_\ell)$. Then, the $100\tau\%$ Bayesian prediction bounds for $Y_{s:\ell:N}$ are obtained by solving the following equations with respect to t:*

$$c(N,s)I^{-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\int_{\theta\in\Theta}\eta(\theta;\underline{\mathbf{x}})\exp[-(\zeta(\theta;\underline{\mathbf{x}})+T_{q,s}\lambda(t,\theta))]d\theta = \begin{cases}(1+\tau)/2,\\(1-\tau)/2,\end{cases} \tag{3.7}$$

where $1 \le s \le \ell$, $\eta(\theta;\underline{\mathbf{x}})$, $\zeta(\theta;\underline{\mathbf{x}})$ and I are given as (2.6), (2.7) *and* (2.8)*, respectively.*

Proof Upon substituting (2.1) and (2.2) in (3.6), we obtain the marginal density function of $Y_{s:\ell:N}$ to be

$$f(y_s|\underline{\mathbf{x}}) = f_{Y_{s:\ell:N}}(y_s) = c(N,s)\sum_{q=0}^{s-1} c_{q,s-1}\lambda'(y_s,\theta)\exp[-T_{q,s}\lambda(y_s,\theta)]. \tag{3.8}$$

By combining the posterior density in (2.5) with the marginal density in (3.8), and then integrating out the parameter θ, we obtain the Bayesian predictive density function of $Y_{s:\ell:N}$, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, to be

$$\begin{aligned} p(y_s|\underline{\mathbf{x}}) &= \int\limits_{\theta\in\Theta} f(y_s|\underline{\mathbf{x}})\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= c(N,s)I^{-1}\sum_{q=0}^{s-1} c_{q,s-1} \int\limits_{\theta\in\Theta} \lambda'(y_s,\theta)\eta(\theta;\underline{\mathbf{x}})\exp[-(\zeta(\theta;\underline{\mathbf{x}})+T_{q,s}\lambda(y_s,\theta))]d\theta. \end{aligned} \tag{3.9}$$

From which, we simply obtain the predictive SF of $Y_{s:\ell:N}$, given $X_{1:m:n} = x_1, \ldots, X_{m:m:n} = x_m$, as

$$\begin{aligned} P(Y_{s:\ell:N} > t|\underline{\mathbf{x}}) &= \int\limits_{t}^{\infty} p(y_s|\underline{\mathbf{x}})dy_s \\ &= c(N,s)I^{-1}\sum_{q=0}^{s-1} \frac{c_{q,s-1}}{T_{q,s}} \int\limits_{\theta\in\Theta} \eta(\theta;\underline{\mathbf{x}})\exp[-(\zeta(\theta;\underline{x})+T_{q,s}\lambda(t,\theta))]d\theta. \end{aligned} \tag{3.10}$$

Consequently, the $100\tau\%$ Bayesian prediction bounds for $Y_{s:\ell:N}$ based on the first sample are obtained by solving the following equations with respect to t:

$$P(Y_{s:\ell:N} > t|\underline{\mathbf{x}}) = \begin{cases} (1+\tau)/2, \\ (1-\tau)/2, \end{cases}$$

where $P(X_{j:R_k} > t|\underline{\mathbf{x}})$ is as in (3.10). □

Corollary 3.4 *In the case when the future sample is a complete sample (i.e., $S_1 = S_2 = \cdots = S_\ell = 0$ and $\ell = N$), then the predictive SF of $Y_{s:N}$ becomes*

$$P(Y_{s:N} > t|\underline{\mathbf{x}}) = c(N,s)I^{-1}\sum_{q=0}^{s-1} \frac{c_{q,s-1}}{T_{q,s}} \int\limits_{\theta\in\Theta} \eta(\theta;\underline{\mathbf{x}})\exp[-(\zeta(\theta;\underline{\mathbf{x}})+T_{q,s}\lambda(t,\theta))]d\theta, \tag{3.11}$$

where $1 \le s \le N$, $c(N,s) = \frac{N!}{(N-s)!}$, $c_{q,s-1} = \frac{(-1)^q}{q!(s-q-1)!}$, $T_{q,s} = N - s + q + 1$, *and*

$$I = \int\limits_{\theta\in\Theta} \eta(\theta;\underline{\mathbf{x}})\exp[-\zeta(\theta;\underline{x})]d\theta.$$

Corollary 3.5 *In the case when the observed sample is a right Type-II censored (i.e.,* $R_i = 0,\ 1 \le i \le m-1$ *and* $R_m = n-m$*), and the future sample is complete sample (i.e.,* $S_1 = S_2 = \cdots = S_\ell = 0$ *and* $\ell = N$*), the predictive SF of* $Y_{s:N}$ *reduces to expression* (26) *of* AL-Hussaini (1999).

4 Illustrative examples

In this section, we discuss the Bayesian prediction problems for exponential(θ) model when θ is unknown, and Pareto(α, β), Weibull(α, β) and Burr Type-XII(α, β) models when both α and β are unknown, as illustrative examples.

4.1 Exponential(θ) model

The distribution function in this case is

$$F(x|\theta) = 1 - \exp(-\theta x), \quad x > 0, \tag{4.1}$$

where $\theta > 0$, and so we have

$$\lambda(x;\theta) = \theta x \quad \text{and} \quad \lambda'(x;\theta) = \theta. \tag{4.2}$$

For the case when θ is unknown, we use the conjugate gamma prior for θ with density

$$\pi(\theta;\delta) = \frac{d^c}{\Gamma(c)}\theta^{c-1}\exp(-\theta d), \tag{4.3}$$

where $\theta > 0$ and $c, d > 0$, and so we have

$$C(\theta;\delta) = \theta^{c-1}, \quad D(\theta;\delta) = \theta d \quad \text{and} \quad \delta = (c, d). \tag{4.4}$$

4.1.1 One-sample Bayesian prediction

The predictive SF of $X_{j:R_k}$ in this case is given by

$$\begin{aligned} P(X_{j:R_k} > t|\underline{\mathbf{x}}) &= D_j(R_k) I^{-1} \sum_{w=0}^{j-1} \frac{c_w(j)}{(R_k - j + w + 1)} \int_0^\infty \theta^{m+c-1} \\ &\quad \times \exp\left[-\theta\left\{\sum_{i=1}^m (R_i+1)x_i + (R_k - j + w + 1)(t - x_k) + d\right\}\right] d\theta \\ &= D_j(R_k) I^{*-1} \sum_{w=0}^{j-1} \frac{c_w(j)}{(R_k - j + w + 1)} \\ &\quad \times \left(\sum_{i=1}^m (R_i+1)x_i + (R_k - j + w + 1)(t - x_k) + d\right)^{-(m+c)}, \end{aligned} \tag{4.5}$$

where

$$I^* = \left(\sum_{i=1}^{m}(R_i + 1)x_i + d\right)^{-(m+c)}.$$

Corollary 4.1 *In the case when the observed sample is Type-II right censored, the predictive SF of $X_{m+j:n}$ agrees with the expression (4.1) of Dunsmore (1974).*

4.1.2 Two-sample Bayesian prediction

The predictive SF of $Y_{s:\ell:N}$ in this case is given by

$$\begin{aligned} P(Y_{s:N} > t|\underline{\mathbf{x}}) &= c(N, s)I^{-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\int_0^\infty \alpha^{m+c-1} \\ &\quad \times \exp\left[-\alpha\left\{\sum_{i=1}^{m}(R_i + 1)x_i + T_{q,s}t + d\right\}\right]d\alpha \\ &= c(N, s)I^{*-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\left(\sum_{i=1}^{m}(R_i+1)x_i+T_{q,s}t+d\right)^{-(m+c)}, \end{aligned} \tag{4.6}$$

where

$$I^* = \left(\sum_{i=1}^{m}(R_i + 1)x_i + d\right)^{-(m+c)}.$$

Corollary 4.2 *In the case when the future sample is a complete sample, the predictive SF of $Y_{s:N}$ becomes*

$$P(Y_{s:N} > t|\underline{\mathbf{x}}) = c(N, s)I^{*-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\left(\sum_{i=1}^{m}(R_i+1)x_i+T_{q,s}t+d\right)^{-(m+c)}, \tag{4.7}$$

where $1 \le s \le N$, $c(N, s) = \frac{N!}{(N-s)!}$, $c_{q,s-1} = \frac{(-1)^q}{q!(s-q-1)!}$, $T_{q,s} = N - s + q + 1$, and

$$I^* = \left(\sum_{i=1}^{m}(R_i + 1)x_i + d\right)^{-(m+c)}.$$

Corollary 4.3 *In the case when the observed sample is Type-II right censored, and the future sample is complete sample, the predictive SF of $Y_{s:N}$ agrees with the expression (2.1) of Dunsmore (1974).*

4.2 Pareto(α, β) model

The distribution function of this model is given by

$$F(x|\theta) = 1 - \exp[-\alpha \ln(x/\beta)], \quad x > \beta, \tag{4.8}$$

where $\theta = (\alpha, \beta)$, $\alpha > 0$ and $\beta > 0$, and so we have

$$\lambda(x; \alpha, \beta) = \alpha \ln(x/\beta) \quad \text{and} \quad \lambda'(x; \alpha, \beta) = \frac{\alpha}{x}. \tag{4.9}$$

Under the assumption that both of the parameters α and β are unknown, we may consider the joint prior density function for α and β which was suggested by Lwin (1972) and generalized by Arnold and Press (1989). The generalized Lwin prior or the power-gamma prior is given by

$$\pi(\alpha, \beta) = \pi_1(\alpha)\pi_2(\beta|\alpha),$$

where

$$\pi_1(\alpha) = \frac{d^c}{\Gamma(c)}\alpha^{c-1} \exp[-d\alpha], \tag{4.10}$$

and

$$\pi_2(\beta|\alpha) = \alpha a \beta^{\alpha a-1} b^{-\alpha a}. \tag{4.11}$$

Then, we have

$$C(\theta; \delta) = \alpha^c \beta^{-1} \quad \text{and} \quad D(\theta; \delta) = \alpha \left[d + a \ln \frac{b}{\beta} \right], \tag{4.12}$$

where $\delta = (a, b, c, d)$, $a, b, c, d > 0, \theta = (\alpha, \beta), \alpha > 0$ and $0 < \beta < b$.

4.2.1 One-sample Bayesian prediction

The predictive SF of $X_{j:R_k}$ in this case is given by

$$\begin{aligned} P(X_{j:R_k} > t|\underline{\mathbf{x}}) = D_j(R_k) I^{-1} \sum_{w=0}^{j-1} \frac{c_w(j)}{(R_k - j + w + 1)} \int_0^L \int_0^\infty \alpha^{(m+c)} \beta^{-1} \left(\prod_{i=1}^m \frac{1}{x_i} \right) \\ \times \exp\left[-\alpha \left\{ \sum_{i=1}^m (R_i + 1) \ln\left(\frac{x_i}{\beta}\right) + a \ln\left(\frac{b}{\beta}\right) + d \right.\right. \\ \left.\left. + \ (R_k - j + w + 1) \ln\left(\frac{t}{x_k}\right) \right\} \right] d\alpha d\beta \end{aligned}$$

$$= D_j(R_k) I^{*-1} \sum_{w=0}^{j-1} \frac{c_w(j)}{\text{ł}(R_k - j + w + 1)} \times \left(\sum_{i=1}^{m} (R_i + 1) \ln \left(\frac{x_i}{L} \right) + a \ln \frac{b}{L} + d + (R_k - j + w + 1) \ln \left(\frac{t}{x_k} \right) \right)^{-(m+c)}, \tag{4.13}$$

where

$$I^* = \left(\sum_{i=1}^{m} (R_i + 1) \ln \left(\frac{x_i}{L} \right) + a \ln \frac{b}{L} + d \right)^{-(m+c)},$$

and $L = \min(x_1, b)$.

When $k = m$, (4.13) agrees with the corresponding expression of Amin (2008).

Corollary 4.4 *In the case when the observed sample is Type-II right censored, the predictive SF of* $X_{m+j:n}$ *reduces to expression* (21) *of* Nigm and Hamdy (1987) *and expression* (66) *of* AL-Hussaini (1999).

4.2.2 Two-sample Bayesian prediction

The predictive SF of $Y_{s:\ell:N}$ in this case is given by

$$P(Y_{s:\ell:N} > t | \underline{\mathbf{x}}) = c(N, s) I^{-1} \sum_{q=0}^{s-1} \frac{c_{q,s-1}}{T_{q,s}} \int_0^L \int_0^\infty \alpha^{(m+c)} \beta^{-1} \left(\prod_{i=1}^{m} \frac{1}{x_i} \right) \times \exp \left[-\alpha \left\{ \sum_{i=1}^{m} (R_i + 1) \ln \left(\frac{x_i}{\beta} \right) + a \ln \frac{b}{\beta} + d + T_{q,s} \ln \left(\frac{t}{\beta} \right) \right\} \right] d\alpha d\beta$$

$$= c(N, s) I^{*-1} \sum_{q=0}^{s-1} \frac{c_{q,s-1}}{T_{q,s}(a + n + T_{q,s})} \times \left(\sum_{i=1}^{m} (R_i + 1) \ln \left(\frac{x_i}{L} \right) + a \ln \frac{b}{L} + d + T_{q,s} \ln \left(\frac{t}{L} \right) \right)^{-(m+c)}, \tag{4.14}$$

where

$$I^* = \frac{1}{(a+n)} \left(\sum_{i=1}^{m} (R_i + 1) \ln \left(\frac{x_i}{L} \right) + a \ln \frac{b}{L} + d \right)^{-(m+c)}.$$

Corollary 4.5 *In the case when the future sample is a complete sample, the predictive SF of $Y_{s:N}$ reduces to expression* (63) *obtained by* Ali Mousa (2001).

Corollary 4.6 *In the case when the observed sample is Type-II right censored, and the future sample is complete sample, the predictive SF of $Y_{s:N}$ becomes*

$$P(Y_{s:N} > t|\underline{\mathbf{x}}) = c(N,s)I^{*-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}(a+n+T_{q,s})} \times\left(\sum_{i=1}^{m}\ln\left(\frac{x_i}{L}\right)+(n-m)\ln\left(\frac{x_i}{L}\right)+a\ln\frac{b}{L} +d+T_{q,s}\ln\left(\frac{t}{L}\right)\right)^{-(m+c)}, \tag{4.15}$$

where $1\leq s\leq N$, $c(N,s)=\frac{N!}{(N-s)!}$, $c_{q,s-1}=\frac{(-1)^q}{q!(s-q-1)!}$, $T_{q,s}=N-s+q+1$, *and*

$$I^* = \frac{1}{(a+n)}\left(\sum_{i=1}^{m}\ln\left(\frac{x_i}{L}\right)+(n-m)\ln\left(\frac{x_i}{L}\right)+a\ln\frac{b}{L}+d\right)^{-(m+c)}.$$

4.3 Weibull(α,β) model

The distribution function of this model is given by

$$F(x|\theta) = 1-\exp[-\alpha x^{\beta})],\quad x>0, \tag{4.16}$$

where $\theta=(\alpha,\beta)$, $\alpha>0$ and $\beta>0$, and so we have

$$\lambda(x;\alpha,\beta)=\alpha x^{\beta}\quad\text{and}\quad\lambda'(x;\alpha,\beta)=\alpha\beta x^{\beta-1}. \tag{4.17}$$

Suppose both α and β are unknown. In this case, we will use the prior density function suggested by Nigm (1989), given by

$$\pi(\alpha,\beta)=\pi_1(\alpha)\pi_2(\beta|\alpha),$$

where both π_1 and π_2 are gamma distributions. Then, the joint prior density function of α and β is

$$\pi(\alpha,\beta)\propto\beta^{2a}\alpha^{a+\frac{c}{\phi(\beta)}}\exp[-(b\beta+d\alpha\psi(\beta))],$$

where $a>-1$, $b,c,d>0$, $\alpha,\beta>0$, $\phi(\beta)$ and $\psi(\beta)$ are increasing functions of β. So,

$$C(\theta;\delta)=\beta^{2a}\alpha^{a+\frac{c}{\phi(\beta)}}\quad\text{and}\quad D(\theta;\delta)=b\beta+d\alpha\psi(\beta) \tag{4.18}$$

where $\delta=(a,b,c,d)$ and $\theta=(\alpha,\beta)$.

4.3.1 One-sample Bayesian prediction

The predictive SF of $X_{j:R_k}$ in this case is given by

$$\begin{aligned}
&P(X_{j:R_k} > t|\underline{\mathbf{x}}) \\
&= D_j(R_k) I^{-1} \sum_{w=0}^{j-1} \frac{c_w(j)}{(R_k - j + w + 1)} \int_0^\infty \int_0^\infty \alpha^{(m+a+\frac{c}{\phi(\beta)})} \beta^{m+2a} \left(\prod_{i=1}^m x_i^{\beta-1} \right) \exp[-b\beta] \\
&\quad \times \exp\left[-\alpha \left\{ \sum_{i=1}^m (R_i + 1) x_i^\beta + d\psi(\beta) + (R_k - j + w + 1) \left(t^\beta - x_k^\beta \right) \right\} \right] d\alpha d\beta \\
&= D_j(R_k) I^{-1} \sum_{w=0}^{j-1} \frac{c_w(j)}{(R_k - j + w + 1)} \int_0^\infty \Gamma(M) \beta^{m+2a} \left(\prod_{i=1}^m x_i^{\beta-1} \right) \exp[-b\beta] \\
&\quad \times \left(\sum_{i=1}^m (R_i + 1) x_i^\beta + d\psi(\beta) + (R_k - j + w + 1)(t^\beta - x_k^\beta) \right)^{-M} d\beta, \qquad (4.19)
\end{aligned}$$

where $M = m + a + \frac{c}{\phi(\beta)} + 1$ and

$$I = \int_0^\infty \Gamma(M) \beta^{m+2a} \left(\prod_{i=1}^m x_i^{\beta-1} \right) \exp[-b\beta] \left(\sum_{i=1}^m (R_i + 1) x_i^\beta + d\psi(\beta) \right)^{-M} d\beta.$$

Corollary 4.7 *In the case when the observed sample is Type-II right censored, the predictive SF of $X_{m+j:n}$ reduces to expression* (44) *of* AL-Hussaini (1999). *It also agrees with expression* (18) *of* Nigm (1989) *except for the term $\Gamma(M)$ that must be inside the two integrals of $P(X_{j:R_k} > t|\underline{x})$ and I.*

4.3.2 Two-sample Bayesian prediction

The predictive SF of $Y_{s:\ell:N}$ in this case is given by

$$\begin{aligned}
P(Y_{s:N} > t|\underline{\mathbf{x}}) &= c(N, s) I^{-1} \sum_{q=0}^{s-1} \frac{c_{q,s-1}}{T_{q,s}} \int_0^\infty \int_0^\infty \alpha^{(m+a+\frac{c}{\phi(\beta)})} \beta^{m+2a} \left(\prod_{i=1}^m x_i^{\beta-1} \right) \exp[-b\beta] \\
&\quad \times \exp\left[-\alpha \left\{ \sum_{i=1}^m (R_i + 1) x_i^\beta + d\psi(\beta) + T_{q,s} t^\beta \right\} \right] d\alpha d\beta \\
&= c(N, s) I^{-1} \sum_{q=0}^{s-1} \frac{c_{q,s-1}}{T_{q,s}} \int_0^\infty \Gamma(M) \beta^{m+2a} \left(\prod_{i=1}^m x_i^{\beta-1} \right) \exp[-b\beta] \\
&\quad \times \left(\sum_{i=1}^m (R_i + 1) x_i^\beta + d\psi(\beta) + T_{q,s} t^\beta \right)^{-M} d\beta, \qquad (4.20)
\end{aligned}$$

where

$$I=\int_{0}^{\infty}\Gamma(M)\beta^{m+2a}\left(\prod_{i=1}^{m}x_i^{\beta-1}\right)\exp[-b\beta]\left(\sum_{i=1}^{m}(R_i+1)x_i^{\beta}+d\psi(\beta)\right)^{-M}d\beta.$$

Corollary 4.8 *In the case when the future sample is a complete sample, the predictive SF of $Y_{s:N}$ becomes*

$$P(Y_{s:N}>t|\underline{\mathbf{x}})=c(N,s)I^{-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\int_{0}^{\infty}\Gamma(M)\beta^{m+2a}\left(\prod_{i=1}^{m}x_i^{\beta-1}\right)\exp[-b\beta]$$
$$\times\left(\sum_{i=1}^{m}(R_i+1)x_i^{\beta}+d\psi(\beta)+T_{q,s}t^{\beta}\right)^{-M}d\beta,\qquad(4.21)$$

where $1\le s\le N$, $c(N,s)=\frac{N!}{(N-s)!}$, $c_{q,s-1}=\frac{(-1)^q}{q!(s-q-1)!}$, $T_{q,s}=N-s+q+1$, and

$$I=\int_{0}^{\infty}\Gamma(M)\beta^{m+2a}\left(\prod_{i=1}^{m}x_i^{\beta-1}\right)\exp[-b\beta]\left(\sum_{i=1}^{m}(R_i+1)x_i^{\beta}+d\psi(\beta)\right)^{-M}d\beta.$$

Corollary 4.9 *In the case when the observed sample is Type-II right censored, and the future sample is complete sample, the predictive SF of $Y_{s:N}$ agrees with expression* (9) *of* Nigm (1989) *for the case $s=1$, except for the term $\Gamma(M)$ that must be inside the two integrals of $P(Y_{s:N}>t|\underline{\mathbf{x}})$ and I.*

4.4 Burr type-XII(α, β) model

The distribution function in this case is given by

$$F(x|\theta)=1-[1+x^{\beta}]^{-\alpha}=1-\exp[-\alpha\ln(1+x^{\beta})],\qquad(4.22)$$

where $\theta=(\alpha,\beta)$, $\alpha>0$, $\beta>0$ and $x>0$, and so we have

$$\lambda(x;\alpha,\beta)=\alpha\ln(1+x^{\beta})\quad\text{and}\quad\lambda'(x;\alpha,\beta)=\frac{\alpha\beta x^{\beta-1}}{1+x^{\beta}}.\qquad(4.23)$$

Suppose both α and β are unknown. We will use in this case the bivariate prior density function suggested by AL-Hussaini and Jaheen (1992), given by

$$\pi(\alpha,\beta)\propto\beta^{a+d}\alpha^{a}\exp\left[-\frac{\beta}{b}\right]\exp\left[-\alpha\left(\frac{\beta}{d}\right)\right],$$

where $a > -1$, $b, c, d > 0$ and $\alpha, \beta > 0$. So,

$$C(\theta;\delta) = \beta^{a+d}\alpha^{a}\exp\left[-\frac{\beta}{b}\right] \quad \text{and} \quad D(\theta;\delta) = \alpha\left(\frac{\beta}{d}\right), \tag{4.24}$$

where $\delta = (a, b, c, d)$ and $\theta = (\alpha, \beta)$.

4.4.1 One-sample Bayesian prediction

The predictive SF of $X_{j:R_k}$ in this case is given by

$$\begin{aligned}
P(X_{j:R_k} > t|\underline{\mathbf{x}}) &= D_j(R_k)I^{-1}\sum_{w=0}^{j-1}\frac{c_w(j)}{(R_k - j + w + 1)} \\
&\quad \times \int_0^\infty\int_0^\infty \alpha^{(m+a)}\beta^{m+a+d}\left(\prod_{i=1}^{m}\frac{x_i^{\beta-1}}{1+x_i^{\beta}}\right)\exp\left[-\frac{\beta}{b}\right] \\
&\quad \times \exp\left[-\alpha\left\{\sum_{i=1}^{m}(R_i+1)\ln(1+x_i^{\beta}) + \frac{\beta}{d}\right.\right. \\
&\qquad \left.\left. +(R_k - j + w + 1)\ln\left(\frac{1+t^{\beta}}{1+x_k^{\beta}}\right)\right\}\right] d\alpha d\beta \\
&= D_j(R_k)I^{*-1}\sum_{w=0}^{j-1}\frac{c_w(j)}{(R_k - j + w + 1)} \\
&\quad \times \int_0^\infty \beta^{m+a+d}\left(\prod_{i=1}^{m}\frac{x_i^{\beta-1}}{1+x_i^{\beta}}\right)\exp\left[-\frac{\beta}{b}\right] \\
&\quad \times \left(\sum_{i=1}^{m}(R_i+1)\ln\left(1+x_i^{\beta}\right) + \frac{\beta}{d}\right. \\
&\qquad \left. + (R_k - j + w + 1)\ln\left(\frac{1+t^{\beta}}{1+x_k^{\beta}}\right)\right)^{-M_2} d\beta,
\end{aligned} \tag{4.25}$$

where $M_2 = m + a + 1$ and

$$I^{*} = \int_0^\infty \beta^{m+a+d}\left(\prod_{i=1}^{m}\frac{x_i^{\beta-1}}{1+x_i^{\beta}}\right)\exp\left[-\frac{\beta}{b}\right]\left(\sum_{i=1}^{m}(R_i+1)\ln\left(1+x_i^{\beta}\right) + \frac{\beta}{d}\right)^{-M_2} d\beta.$$

Corollary 4.10 *In the case when the observed sample is Type-II right censored, the predictive SF of $X_{m+j:n}$ becomes expression* (31) *of* AL-Hussaini and Jaheen (1992).

4.4.2 Two-sample Bayesian prediction

The predictive SF of $Y_{s:\ell:N}$ in this case is given by

$$\begin{aligned} P(Y_{s:N}>t|\underline{\mathbf{x}}) &= c(N,s)I^{-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\int_0^\infty\int_0^\infty \alpha^{(m+a)}\beta^{m+a+d}\left(\prod_{i=1}^m\frac{x_i^{\beta-1}}{1+x_i^\beta}\right)\exp\left[-\frac{\beta}{b}\right] \\ &\quad\times\exp\left[-\alpha\left\{\sum_{i=1}^m(R_i+1)\ln\left(1+x_i^\beta\right)\right.\right. \\ &\quad\left.\left.+\frac{\beta}{d}+T_{q,s}\ln(1+t^\beta)\right\}\right]d\alpha d\beta \\ &= c(N,s)I^{*-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\int_0^\infty\beta^{m+a+d}\left(\prod_{i=1}^m\frac{x_i^{\beta-1}}{1+x_i^\beta}\right)\exp\left[-\frac{\beta}{b}\right] \\ &\quad\times\left(\sum_{i=1}^m(R_i+1)\ln\left(1+x_i^\beta\right)+\frac{\beta}{d}+T_{q,s}\ln(1+t^\beta)\right)^{-M_2}d\beta, \end{aligned} \tag{4.26}$$

where

$$I^*=\int_0^\infty\beta^{m+a+d}\left(\prod_{i=1}^m\frac{x_i^{\beta-1}}{1+x_i^\beta}\right)\exp\left[-\frac{\beta}{b}\right]\left(\sum_{i=1}^m(R_i+1)\ln\left(1+x_i^\beta\right)+\frac{\beta}{d}\right)^{-M_2}d\beta.$$

Corollary 4.11 *In the case when the future sample is a complete sample, the predictive SF of $Y_{s:N}$ becomes expression* (26) *of* Ali Mousa and Jaheen (2002)*.*

Corollary 4.12 *In the case when the observed sample is Type-II right censored, and the future sample is complete sample, the predictive SF of $Y_{s:N}$ becomes*

$$\begin{aligned} P(Y_{s:N}>t|\underline{\mathbf{x}}) &= c(N,s)I^{*-1}\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{T_{q,s}}\int_0^\infty\beta^{m+a+d}\left(\prod_{i=1}^m\frac{x_i^{\beta-1}}{1+x_i^\beta}\right)\exp\left[-\frac{\beta}{b}\right] \\ &\quad\times\left(\sum_{i=1}^m\ln\left(1+x_i^\beta\right)+(n-m)\ln\left(1+x_m^\beta\right)\right. \\ &\quad\left.+\frac{\beta}{d}+T_{q,s}\ln(1+t^\beta)\right)^{-M_2}d\beta, \end{aligned} \tag{4.27}$$

where $1\le s\le N$*,*

$$c(N,s)=\frac{N!}{(N-s)!},\ c_{q,s-1}=\frac{(-1)^q}{q!(s-q-1)!},\ T_{q,s}=N-s+q+1,$$

and

$$I^* = \int_0^\infty \beta^{m+a+d} \left(\prod_{i=1}^{m} \frac{x_i^{\beta-1}}{1+x_i^\beta} \right) \exp\left[-\frac{\beta}{b} \right]$$
$$\times \left(\sum_{i=1}^{m} \ln\left(1+x_i^\beta\right) + (n-m)\ln\left(1+x_m^\beta\right) + \frac{\beta}{d} \right)^{-M_2} d\beta.$$

5 Numerical illustration

To illustrate the inferential procedures developed in the preceding sections, we present a numerical study for two distributions. The first distribution is exponential(θ) distribution when θ is unknown, while the second one is Pareto(α,β) distribution when α and β are both unknown.

5.1 Example 1

In this example, we illustrate the prediction results for exponential(θ) distribution when θ is unknown. We generated a progressively Type-II right censored sample of size $m = 15$ from a sample of size $n = 25$ with the progressive censoring scheme $\underline{\mathbf{R}} = (1, 2, 1, 0, 0, 1, 2, 0, 0, 0, 1, 0, 0, 0, 2)$, by using the algorithms of Balakrishnan and Sandhu (1995). The generated progressively Type-II censored sample from exponential distribution (with $\theta = 0.5$) is as follows:

0.01, 0.21, 0.77, 0.86, 0.94, 1.05, 1.38, 1.39, 1.83, 1.93, 2.46, 2.64, 6.44, 8.50 and 10.04.

We assume these data to have come from exponential(θ) distribution, where θ is unknown. We then used the results presented earlier in Sect. 4.1 to compute one- and two-sample Bayesian prediction bounds.

Based on the generated progressively Type-II censored sample, Table 1 presents the 95% Bayesian prediction bounds of $X_{j:R_k}$ for four different choices of hyperparameters (c, d).

Based on the generated progressively Type-II censored sample, Table 2 presents the 95% Bayesian prediction intervals of $Y_{s:\ell:N}$ from the future progressively Type-II censored sample of size $\ell = 10$ from a sample of size $N = 20$ with progressive censoring scheme $\underline{\mathbf{S}} = (1, 1, 0, 2, 1, 0, 1, 1, 0, 3)$ for four different choices of hyperparameters (c, d).

5.2 Example 2

In this example, we illustrate the prediction results for Pareto(α,β) distribution when α and β are both unknown. We generated a progressively Type-II right censored sample of size $m = 15$ from a sample of size $n = 25$ with the progressive censoring scheme

Table 1 95% Bayesian prediction intervals of $X_{j:R_k}$ for $j = 1, \ldots, R_k$ and $k = 1, \ldots, m$

(c, d)			(1, 5)		(1, 4)		(1, 6)		(2, 6)	
k	R_k	j	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$
1	1	1	0.13	18.95	0.12	18.69	0.13	19.21	0.12	17.96
2	2	1	0.27	9.68	0.27	9.55	0.27	9.81	0.27	9.18
2	2	2	0.98	23.13	0.97	22.82	0.99	23.44	0.95	21.90
3	1	1	0.89	19.71	0.89	19.46	0.89	19.97	0.88	18.72
6	1	1	1.16	19.99	1.16	19.73	1.17	20.25	1.16	18.99
7	2	1	1.45	10.86	1.45	10.73	1.45	10.99	1.44	10.36
7	2	2	2.16	24.30	2.15	23.99	2.17	24.62	2.12	23.07
11	1	1	2.58	21.40	2.57	21.14	2.58	21.66	2.57	20.40
15	2	1	10.10	19.51	10.10	19.38	10.10	19.64	10.10	19.02
15	2	2	10.81	32.96	10.80	32.65	10.82	33.27	10.78	31.73

Table 2 95% Bayesian prediction intervals of $Y_{s:\ell:N}$ for $s = 1, 2, \ldots, \ell$

(c, d)	(1, 5)		(1, 4)		(1, 6)		(2, 6)	
s	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$
1	0.01	0.95	0.01	0.93	0.01	0.96	0.01	0.90
2	0.06	1.55	0.06	1.53	0.06	1.57	0.05	1.47
3	0.15	2.18	0.15	2.15	0.15	2.21	0.14	2.06
4	0.27	2.82	0.27	2.78	0.28	2.86	0.26	2.66
5	0.44	3.63	0.43	3.58	0.44	3.68	0.42	3.43
6	0.64	4.61	0.63	4.55	0.64	4.67	0.61	4.35
7	0.87	5.67	0.86	5.59	0.88	5.75	0.83	5.35
8	1.16	7.08	1.15	6.98	1.18	7.17	1.11	6.67
9	1.55	9.14	1.53	9.01	1.57	9.26	1.49	8.61
10	2.04	11.69	2.01	11.53	2.07	11.85	1.96	11.02

$\underline{\mathbf{R}} = (1, 0, 0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 0, 0, 3)$, by using the algorithms of Balakrishnan and Sandhu (1995). The generated progressively Type-II censored sample from Pareto distribution (with $\alpha = 3$, $\beta = 6$) is as follows:

6.11, 6.14, 6.36, 6.36, 6.48, 6.51, 7.00, 7.10, 7.53, 7.54, 7.67, 7.71, 8.24, 8.88 and 9.38.

We assume these data to have come from Pareto(α, β) distribution, where both parameters α and β are unknown. We then used the results presented earlier in Sect. 4.2 to compute one- and two-sample Bayesian prediction bounds.

Based on the generated progressively Type-II censored sample, Table 3 presents the 95% Bayesian prediction bounds of $X_{j:R_k}$ for four different choices of hyperparameters (a, b, c, d).

Table 3 95% Bayesian prediction intervals of $X_{j:R_k}$ for $j = 1, \ldots, R_k$ and $k = 1, \ldots, m$

(a, b, c, d)			$(1, 7.5, 4, 1)$		$(1, 9, 5, 1)$		$(1, 6, 3, 1)$		$(1, 9, 3, 1)$	
k	R_k	j	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$	$L_{X_{j:R_k}}$	$U_{X_{j:R_k}}$
1	1	1	6.15	21.14	6.15	20.49	6.16	24.07	6.16	23.78
6	3	1	6.53	9.85	6.53	9.75	6.53	10.29	6.53	10.25
6	3	2	6.71	14.66	6.71	14.35	6.73	15.97	6.73	15.85
6	3	3	7.21	33.98	7.20	32.54	7.28	40.49	7.28	39.85
10	3	1	7.55	11.40	7.55	11.28	7.56	11.90	7.56	11.85
10	3	2	7.76	16.96	7.76	16.60	7.78	18.48	7.78	18.33
10	3	3	8.34	39.31	8.33	37.64	8.43	46.84	8.42	46.10
15	3	1	9.40	14.18	9.40	14.04	9.40	14.81	9.40	14.75
15	3	2	9.66	21.10	9.65	20.66	9.69	22.99	9.68	22.81
15	3	3	10.38	48.91	10.36	46.84	10.48	58.28	10.47	57.36

Table 4 95% Bayesian prediction intervals of $X_{j:R_k}$ for $j = 1, \ldots, R_k$ and $k = 1, \ldots, m$

(a, b, c, d)	$(1, 7.5, 4, 1)$		$(1, 9, 5, 1)$		$(1, 6, 3, 1)$		$(1, 9, 3, 1)$	
s	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$	$L_{Y_{s:\ell:N}}$	$U_{Y_{s:\ell:N}}$
1	6.06	6.43	6.06	6.42	5.95	6.35	6.05	6.46
2	6.07	6.70	6.07	6.69	5.96	6.65	6.06	6.76
3	6.09	6.97	6.10	6.95	5.99	6.95	6.05	7.06
4	6.11	7.29	6.11	7.25	6.01	7.29	6.11	7.41
5	6.20	7.78	6.20	7.73	6.10	7.85	6.21	7.97
6	6.30	8.34	6.30	8.27	6.21	8.48	6.32	8.60
7	6.42	9.01	6.42	8.91	6.34	9.24	6.45	9.36
8	6.58	10.03	6.57	9.89	6.51	10.40	6.62	10.53
9	6.76	11.40	6.75	11.20	6.71	11.98	6.82	12.12
10	7.05	14.44	7.03	14.09	7.02	15.58	7.13	15.72

Based on the generated progressively Type-II censored sample, Table 4 presents the 95% Bayesian prediction intervals of $Y_{s:\ell:N}$ from the future progressively Type-II censored sample of size $\ell = 10$ from a sample of size $N = 20$ with progressive censoring scheme $\underline{\mathbf{S}} = (1, 1, 0, 2, 1, 0, 1, 1, 0, 3)$ for four different choices of hyperparameters (a, b, c, d).

6 Concluding remarks

1. From Tables 1 and 3, we notice that, in the one-sample case, the lower bounds are relatively insensitive to the assumed values to hyperparameters while the upper bounds are quite sensitive.

2. From Tables 2 and 4, we notice that, in the two-sample case, the lower bounds and the upper bounds are relatively insensitive to the assumed values to hyperparameters.
3. If the vector of hyperparameters δ is unknown, the empirical Bayes approach could be used in estimating such prior parameters based on past samples; see, for example, Maritz and Lwin (1989). Alternatively, one could use the hierarchical Bayesian method in which some suitable prior for δ could be proposed; see, for example, Geisser (1990); Bernardo and Smith (1994). Work in these directions are currently under progress and we hope to report these findings in a future paper.

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