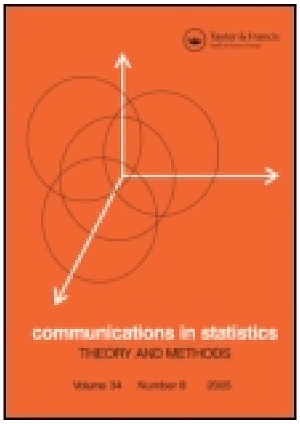

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N. Balakrishnan [a] & A. R. Shafay [b]

[a] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada

[b] Department of Mathematics, Fayoum University, Fayoum, Egypt

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One- and Two-Sample Bayesian Prediction Intervals Based on Type-II Hybrid Censored Data

N. BALAKRISHNAN[1] AND A. R. SHAFAY[2]

[1]Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada
[2]Department of Mathematics, Fayoum University, Fayoum, Egypt

In this article, one- and two-sample Bayesian prediction intervals based on Type-II hybrid censored data are derived. For the illustration of the developed results, the Exponential(θ) and Pareto(α, β) distributions are used as examples. One-sample Bayesian predictive survival function can not be obtained in closed form. Gibbs sampling procedure is therefore used to draw Markov Chain Monte Carlo (MCMC) samples, and they are in turn used to compute the approximate predictive survival function, and the corresponding numerical results are presented.

Keywords Bayesian prediction; Exponential distribution; Markov Chain Monte Carlo; Order statistics; Pareto distribution; Type-II hybrid censored sample.

Mathematics Subject Classification Primary 62N01; Secondary 62F15.

1. Introduction

In reliability analysis, experiments often terminate before all units on test have failed due to cost and time considerations. In such cases, failure information is available only on part of the sample, and on all units that had not failed, one has only partial information. Such data are said to be censored data. The two most common censoring schemes are Type-I and Type-II censoring schemes. They can be described as follows: Consider n identical units under observation in a life-testing experiment. In the Type-I censoring scheme, the experiment is terminated when a pre-fixed censoring time T arrives. On the other hand, in the Type-II censoring scheme, the experiment gets terminated when a pre-specified number $r \leq n$ of failures occur. In both censoring schemes, some information is lost as only a part of the sample is observed, but they do result in a saving in terms of time and cost. In a Type-I censoring scheme, the termination time is guaranteed but the level of efficiency may be too low or too high due to the uncertainty in the number of complete failures. In a Type-II censoring scheme, the level of efficiency is guaranteed (since the number of

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Address correspondence to A. R. Shafay, Department of Mathematics, Fayoum University, Fayoum, Egypt; E-mail: a_elshafay2010@yahoo.com

failures to be observed is fixed in advance), but the duration of the experiment may end up being too long since the exact time of the rth failure is uncertain. For these reasons, another censoring scheme becomes necessary if both efficiency level and guaranteed termination time are sought. A mixture of Type-I and Type-II censoring schemes, known as hybrid censoring scheme, has been discussed in the literature for this purpose.

Epstein (1954) introduced the Type-I hybrid censoring scheme (Type-I HCS) in which the life-testing experiment is terminated as soon as a pre-specified number r out of n items has failed or a pre-fixed time T has been reached in the life-test. The Type-I HCS has been used as a reliability acceptance test in MIL-STD-781 C (1977). However, the Type-I HCS may result in too few observations in the data. For this reason, Childs et al. (2003) proposed the Type-II hybrid censoring scheme (Type-II HCS) in which the life-testing experiment gets terminated whenever the later of the two stopping rules is reached. The Type-II HCS is preferable to use as it gives a guarantee that the number of observations in the data is at least r, thus resulting in more efficient inferential procedures than under Type-I HCS. A hybrid censoring has been discussed extensively in the reliability literature; see, for example, Epstein (1954), Fairbanks et al. (1982), Chen and Bhattacharyya (1988), Jeong et al. (1996), Gupta and Kundu (1998), Childs et al. (2003), Chandrasekar et al. (2004), Kundu (2007), Lin et al. (2008), Kundu and Banerjee (2008), Park et al. (2008), and Park and Balakrishnan (2009).

In many practical problems, one would wish to use past data for predicting a future observation from the same population. One way to do this is to construct an interval that will contain the future observation with a specified probability, called as a prediction interval. Bayesian prediction bounds for future observations based on Type-I censored data have been discussed by several authors, including Evans and Nigm (1980), AL-Hussaini (1999a), and AL-Hussaini et al. (2001). Several authors have studied the Bayesian prediction based on Type-II censored data; see Dunsmore (1974), Nigm and Hamdy (1987), Nigm (1988, 1989), AL-Hussaini and Jaheen (1995), AL-Hussaini (1999b), and Raqab and Madi (2005). Gupta and Kundu (1998) and Kundu (2007) derived Bayes estimates and the corresponding highest posterior density credible intervals for the unknown parameters of exponential and Weibull distributions based on Type-I hybrid censored data. Draper and Guttman (1987) discussed the two-sample Bayesian prediction of a future observation from an exponential distribution based on Type-I hybrid censored data. Ebrahimi (1992) developed the classical prediction intervals for future failures in the case of exponential distribution under Type-I hybrid censoring.

In this article, we consider a general form for the underlying distribution and a general conjugate prior, and describe a general procedure for determining the Bayesian prediction intervals for future life-lengths based on observed Type-II hybrid censored data. In Sec. 2, we derive the one-sample Bayesian predictive survival function and the one-sample Bayesian prediction bounds for the sth ($r < s \leq n$) observation from a Type-II hybrid censored sample. Next, we derive the two-sample Bayesian predictive survival function and the two-sample Bayesian prediction bounds for the sth observation from a future independent sample when the (observed) informative sample is Type-II hybrid censored and the (unobserved) future sample is the usual order statistics from the same parent distribution. In Sec. 3, we present the results for the exponential (θ) and Pareto(α, β) distributions

as illustrative examples and we use Markov Chain Monte Carlo method to compute the approximate predictive survival function in the one-sample case. Finally, in Sec. 4, we present some numerical results for these examples.

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics from a random sample of size n from an absolutely continuous distribution function $F(x) \equiv F(x \mid \theta)$ with density function $f(x) \equiv f(x \mid \theta)$, where the parameter $\theta \in \Theta$ may be a real vector. Let K denote the number of $X_{i:n}$'s that are at most T. Therefore, under the Type-II hybrid censoring scheme described above, we have one of the two following types of observations:

Case I. $X_{1:n} < \cdots < X_{r:n}$ if $X_{r:n} > T$ with $0 \le K \le r-1$;

Case II. $X_{1:n} < \cdots < X_{K:n}$ if $T \ge X_{r:n}$ with $r \le K \le n$.

Thus, the likelihood function of the Type-II hybrid censored sample is as follows:

Case I.

$$f(\underline{\mathbf{x}}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i)\{1 - F(x_r)\}^{n-r}, \tag{1.1}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$, $x_1 < \cdots < x_r$ and $x_r > T$;

Case II.

$$f(\underline{\mathbf{x}}) = \frac{n!}{(n-K)!} \prod_{i=1}^{K} f(x_i)\{1 - F(T)\}^{n-K}, \tag{1.2}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_K)$ and $x_1 < \cdots < x_K \le T < x_{K+1}$.

Lemma 1.1. *Conditional on $K = k$, the vectors $(X_{1:n}, \ldots, X_{k:n})$ and $(X_{k+1:n}, \ldots, X_{n:n})$ are mutually independent with*

$$(X_{1:n}, \ldots, X_{k:n}) \stackrel{d}{=} (V_{1:k}, \ldots, V_{k:k}),$$
$$(X_{k+1:n}, \ldots, X_{n:n}) \stackrel{d}{=} (W_{1:n-k}, \ldots, W_{n-k:n-k}),$$

where $V_{1:k}, \ldots, V_{k:k}$ are order statistics from an iid sample of size k from $F(x)$ right-truncated at T, and $W_{1:n-k}, \ldots, W_{n-k:n-k}$ are order statistics from an iid sample of size $n-k$ from $F(x)$ left-truncated at T.

For a proof of this result as well as some generalizations of this result, one may refer to Iliopoulos and Balakrishnan (2009).

Let $r < s \le n$, then the conditional density function of $X_{s:n}$, given the Type-II hybrid censored sample, is as follows:

Case I.

$$f(x_s \mid \underline{\mathbf{x}}) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{(F(x_s) - F(x_r))^{s-r-1}(1 - F(x_s))^{n-s} f(x_s)}{(1 - F(x_r))^{n-r}}, \tag{1.3}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$ and $x_s > x_r$;

Case II.

$$f(x_s \mid \underline{\mathbf{x}}) = \frac{1}{P(r \le K \le s-1)} \sum_{k=r}^{s-1} f(x_s \mid \underline{\mathbf{x}}, K=k) P(K=k)$$
$$= \sum_{k=r}^{s-1} \frac{(n-k)!\phi_k(T)}{(s-k-1)!(n-s)!} \frac{(F(x_s)-F(T))^{s-k-1}(1-F(x_s))^{n-s} f(x_s)}{(1-F(T))^{n-k}}, \quad (1.4)$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_K)$, $x_s > T$ and $\phi_k(T) = \frac{P(K=k)}{\sum_{j=r}^{s-1} P(K=j)}$.

Let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{m:m}$ be the order statistics (OS) from a future random sample of size m from the same population. Then the marginal density function of the sth OS $Y_{s:m}$ is given by

$$f_{Y_{s:m}}(y_s) = \frac{m!}{(s-1)!(m-s)!} (F(y_s))^{s-1} (1-F(y_s))^{m-s} f(y_s), \quad (1.5)$$

where $1 \le s \le m$.

Since the survival function (SF) $\overline{F}(x \mid \theta) = 1 - F(x \mid \theta)$ corresponding to any cumulative distribution function (cdf) $F(x \mid \theta)$, $\theta \in \Theta$, can be written in the form

$$\overline{F}(x \mid \theta) = \exp[-\lambda(x;\theta)], \quad (1.6)$$

where $\lambda(x;\theta) = -\ln \overline{F}(x \mid \theta)$, we shall consider the underling population SF to be given by (1.6). Of course, some conditions need to be imposed so that $\overline{F}(x \mid \theta)$ is a valid SF. These conditions are: $\lambda(x;\theta)$ is continuous, monotone increasing and differentiable function, with $\lambda(x;\theta) \to 0$ as $x \to -\infty$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. The probability density function (pdf) corresponding to (1.6) is given by

$$f(x \mid \theta) = \lambda'(x;\theta) \exp[-\lambda(x;\theta)], \quad (1.7)$$

where $\lambda'(x;\theta)$ is the derivative of $\lambda(x;\theta)$ with respect to x.

With an appropriate choice of $\lambda(x;\theta)$ (notice that the derivative of $\lambda(x;\theta)$ with respect to x is the hazard rate function), several distributions that are used in reliability studies can be obtained as special cases. For example, if $\lambda(x;\theta) = \theta x$, we obtain the exponential(θ) distribution. If $\lambda(x;\theta) = -\alpha \ln(\beta/x)$, we obtain the Pareto($\alpha, \beta$) distribution. If $\lambda(x;\theta) = \alpha x^\beta$, we obtain the Weibull($\alpha, \beta$) distribution. The Burr Type XII(α, β) distribution is obtained by taking $\lambda(x;\theta) = \alpha \ln(1 + x^\beta)$. Appropriate conditions need to be imposed on $\lambda(x;\theta)$ to suit the domain on which $\overline{F}(x \mid \theta)$ in (1.6) is defined. For example, if $\overline{F}(x \mid \theta)$ is defined only on the positive half of the real line (as for the exponential, Weibull, and Burr Type XII distributions), then $\lambda(x;\theta) \to 0$ as $x \to 0^+$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. If $\overline{F}(x \mid \theta)$ is defined on (β, ∞) (as in the Pareto distribution), then $\lambda(x;\theta) \to 0$ as $x \to \beta^+$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. The exponential form of the SF in (1.6) provides some flexibility in developing general results as done in the following sections.

2. Bayesian Prediction Intervals

For the Bayesian prediction setup, we need a suitable prior parameter distribution. We consider here a general conjugate prior, suggested by AL-Hussaini (1999b), that

is given by

$$\pi(\theta; \delta) \propto C(\theta; \delta) \exp[-D(\theta; \delta)], \tag{2.1}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution under consideration and δ is the vector of prior parameters. The prior family in (2.1) includes several priors used in the literature as special cases.

2.1. *One-sample Bayesian prediction*

Upon using (1.6) and (1.7) in (1.1) and (1.2), we obtain the likelihood function as follows:

Case I.

$$L(\theta; \underline{\mathbf{x}}) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r} \lambda'(x_i, \theta)\right) \exp\left[-\left\{\sum_{i=1}^{r} \lambda(x_i; \theta) + (n-r)\lambda(x_r; \theta)\right\}\right], \tag{2.2}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$, $x_1 < \cdots < x_r$ and $x_r > T$;

Case II.

$$L(\theta; \underline{\mathbf{x}}) = \frac{n!}{(n-K)!}\left(\prod_{i=1}^{K} \lambda'(x_i, \theta)\right) \exp\left[-\left\{\sum_{i=1}^{K} \lambda(x_i; \theta) + (n-K)\lambda(T; \theta)\right\}\right], \tag{2.3}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_K)$ and $x_1 < \cdots < x_K \leq T < x_{K+1}$.

Similarly, upon substituting (1.6) and (1.7) in (1.3) and (1.4), we obtain the conditional density function of $X_{s:n}$, given the Type-II hybrid censored sample, as follows:

Case I.

$$f(x_s \mid \underline{\mathbf{x}}) = \sum_{w=0}^{s-r-1} C_1 \lambda'(x_s, \theta) g_w(x_s, x_r; \theta), \tag{2.4}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$, $x_s > x_r$, $C_1 = \frac{(-1)^w \binom{s-r-1}{w}(n-r)!}{(s-r-1)!(n-s)!}$ and

$$g_w(x, y; \theta) = \exp[-(n-s+w+1)(\lambda(x; \theta) - \lambda(y; \theta))]; \tag{2.5}$$

Case II.

$$f(x_s \mid \underline{\mathbf{x}}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} C_2 \phi_k(T, \theta) \lambda'(x_s, \theta) g_w(x_s, T; \theta), \tag{2.6}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_K)$, $x_s > T$, $C_2 = \frac{(-1)^w \binom{s-k-1}{w}(n-k)!}{(s-k-1)!(n-s)!}$ and

$$\phi_k(T, \theta) = \frac{\binom{n}{k} \exp[-(n-k)\lambda(T; \theta)](1 - \exp[-\lambda(T; \theta)])^k}{\sum_{j=r}^{s-1} \binom{n}{j} \exp[-(n-j)\lambda(T; \theta)](1 - \exp[-\lambda(T; \theta)])^j}. \tag{2.7}$$

Since the posterior density function is given by

$$\pi^*(\theta \mid \underline{\mathbf{x}}) = I^{-1}\pi(\theta; \delta)L(\theta; \underline{\mathbf{x}}), \tag{2.8}$$

where

$$I = \int_{\theta \in \Theta} \pi(\theta; \delta)L(\theta; \underline{\mathbf{x}})d\theta, \tag{2.9}$$

we obtain from (2.1), (2.2), (2.3), and (2.8) the posterior density function as follows:

Case I.

$$\pi_1^*(\theta \mid \underline{\mathbf{x}}) = I_1^{-1}\eta_1(\theta; \underline{\mathbf{x}}) \exp[-\zeta_1(\theta; \underline{\mathbf{x}})], \tag{2.10}$$

where

$$\eta_1(\theta; \underline{\mathbf{x}}) = C(\theta; \delta) \prod_{i=1}^{r} \lambda'(x_i, \theta),$$

$$\zeta_1(\theta; \underline{\mathbf{x}}) = \sum_{i=1}^{r} \lambda(x_i; \theta) + (n-r)\lambda(x_r; \theta) + D(\theta; \delta),$$

and

$$I_1 = \int_{\theta \in \Theta} \eta_1(\theta; \underline{\mathbf{x}}) \exp[-\zeta_1(\theta; \underline{x})]d\theta;$$

Case II.

$$\pi_2^*(\theta \mid \underline{\mathbf{x}}) = I_2^{-1}\eta_2(\theta; \underline{\mathbf{x}}) \exp[-\zeta_2(\theta; \underline{\mathbf{x}})], \tag{2.11}$$

where

$$\eta_2(\theta; \underline{\mathbf{x}}) = C(\theta; \delta) \prod_{i=1}^{K} \lambda'(x_i, \theta),$$

$$\zeta_2(\theta; \underline{\mathbf{x}}) = \sum_{i=1}^{K} \lambda(x_i; \theta) + (n-K)\lambda(T; \theta) + D(\theta; \delta),$$

and

$$I_2 = \int_{\theta \in \Theta} \eta_2(\theta; \underline{\mathbf{x}}) \exp[-\zeta_2(\theta; \underline{x})]d\theta.$$

Since the Bayesian predictive density function of $X_{s:n}$ is given by

$$p(x_s \mid \underline{\mathbf{x}}) = \int_{\theta\in\Theta} f(x_s \mid \underline{\mathbf{x}})\pi^*(\theta \mid \underline{\mathbf{x}})d\theta, \tag{2.12}$$

upon using (2.4), (2.6), (2.10), and (2.11) in (2.12), we obtain the predictive density function of $X_{s:n}$ as follows:

Case I.

$$p(x_s \mid \underline{\mathbf{x}}) = \sum_{w=0}^{s-r-1} C_1 \int_{\theta\in\Theta} \lambda'(x_s, \theta) g_w(x_s, x_r; \theta)\pi_1^*(\theta \mid \underline{\mathbf{x}})d\theta, \tag{2.13}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$ and $x_s > x_r$;

Case II.

$$p(x_s \mid \underline{\mathbf{x}}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} C_2 \int_{\theta\in\Theta} \lambda'(x_s, \theta)\phi_k(T, \theta) g_w(x_s, T; \theta)\pi_2^*(\theta \mid \underline{\mathbf{x}})d\theta, \tag{2.14}$$

where $\underline{\mathbf{x}} = (x_1, \ldots, x_K)$ and $x_s > T$.

So, the predictive survival function of $X_{s:n}$ is obtained as follows:

Case I.

$$\begin{aligned} P(X_{s:n} > t \mid \underline{\mathbf{x}}) &= \sum_{w=0}^{s-r-1} C_1 \int_t^\infty \int_{\theta\in\Theta} \lambda'(x_s, \theta) g_w(x_s, x_r; \theta)\pi_1^*(\theta \mid \underline{\mathbf{x}})d\theta dx_s \\ &= \sum_{w=0}^{s-r-1} \frac{C_1}{n-s+w+1} \int_{\theta\in\Theta} g_w(t, x_r; \theta)\pi_1^*(\theta \mid \underline{\mathbf{x}})d\theta; \end{aligned} \tag{2.15}$$

Case II.

$$\begin{aligned} P(X_{s:n} > t \mid \underline{\mathbf{x}}) &= \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} C_2 \int_t^\infty \int_{\theta\in\Theta} \lambda'(x_s, \theta)\phi_k(T, \theta) g_w(x_s, T; \theta)\pi_2^*(\theta \mid \underline{\mathbf{x}})d\theta dx_s \\ &= \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_{\theta\in\Theta} \phi_k(T, \theta) g_w(t, T; \theta)\pi_2^*(\theta \mid \underline{\mathbf{x}})d\theta. \end{aligned} \tag{2.16}$$

Then, the $100\tau\%$ Bayesian prediction interval (L, U) for $X_{s:n}$ based on the Type-II hybrid censored sample is obtained by solving the following two equations:

$$P(X_{s:n} > L \mid \underline{\mathbf{x}}) = \frac{1+\tau}{2}, \tag{2.17}$$

and

$$P(X_{s:n} > U \mid \underline{\mathbf{x}}) = \frac{1-\tau}{2}. \tag{2.18}$$

2.2. *Two-sample Bayesian Prediction*

Upon substituting (1.6) and (1.7) in (1.5), we obtain

$$f(y_s \mid \underline{\mathbf{x}}) = \sum_{w=0}^{s-1} C_3 \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)], \tag{2.19}$$

where $1 \le s \le m$ and $C_3 = \frac{(-1)^w \binom{s-1}{w} m!}{(s-1)!(m-s)!}$.

Since the Bayesian predictive density function of $Y_{s:m}$ is given by

$$p(y_s \mid \underline{\mathbf{x}}) = \int_{\theta \in \Theta} f(y_s \mid \underline{\mathbf{x}}) \pi^*(\theta \mid \underline{\mathbf{x}}) d\theta, \tag{2.20}$$

upon substituting (2.10), (2.11), and (2.19) in (2.20), we obtain the Bayesian predictive density function of $Y_{s:m}$ as follows:

Case I.

$$p(y_s \mid \underline{\mathbf{x}}) = \sum_{w=0}^{s-1} C_3 \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi_1^*(\theta \mid \underline{\mathbf{x}}) d\theta; \tag{2.21}$$

Case II.

$$p(y_s \mid \underline{\mathbf{x}}) = \sum_{w=0}^{s-1} C_3 \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi_2^*(\theta \mid \underline{\mathbf{x}}) d\theta. \tag{2.22}$$

From (2.21) and (2.22), we obtain the predictive survival function of $Y_{s:m}$ as follows:

Case I.

$$\begin{aligned} P(Y_{s:m} > t \mid \underline{\mathbf{x}}) &= \sum_{w=0}^{s-1} C_3 \int_t^\infty \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi_1^*(\theta \mid \underline{\mathbf{x}}) d\theta \, dy_s \\ &= \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \int_{\theta \in \Theta} \exp[-(m-s+w+1)\lambda(t, \theta)] \pi_1^*(\theta \mid \underline{\mathbf{x}}) d\theta; \end{aligned} \tag{2.23}$$

Case II.

$$\begin{aligned} P(Y_{s:m} > t \mid \underline{\mathbf{x}}) &= \sum_{w=0}^{s-1} C_3 \int_t^\infty \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi_2^*(\theta \mid \underline{\mathbf{x}}) d\theta \, dy_s \\ &= \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \int_{\theta \in \Theta} \exp[-(m-s+w+1)\lambda(t, \theta)] \pi_2^*(\theta \mid \underline{\mathbf{x}}) d\theta. \end{aligned} \tag{2.24}$$

Consequently, the $100\tau\%$ Bayesian prediction interval (L, U) for $Y_{s:m}$ based on the Type-II hybrid censored sample is obtained by solving the following two equations:

$$P(Y_{s:m} > L \mid \underline{\mathbf{x}}) = \frac{1+\tau}{2}, \tag{2.25}$$

and

$$P(Y_{s:m} > U \mid \underline{\mathbf{x}}) = \frac{1-\tau}{2}. \tag{2.26}$$

3. Illustrative Examples

In this section, we discuss the Bayesian prediction problems for the exponential(θ) distribution when θ is unknown, and Pareto(α, β) distribution when both parameters α and β are unknown, as illustrative examples.

3.1. *Exponential(θ) Model*

The distribution function in this case is

$$F(x \mid \theta) = 1 - \exp(-\theta x), \quad x > 0, \tag{3.1}$$

where $\theta > 0$, and so we have

$$\lambda(x; \theta) = \theta x \quad \text{and} \quad \lambda'(x; \theta) = \theta. \tag{3.2}$$

For the case when θ is unknown, we use the conjugate gamma prior for θ with density

$$\pi(\theta; \delta) = \frac{d^c}{\Gamma(c)} \theta^{c-1} \exp(-\theta d), \quad \theta > 0, \tag{3.3}$$

where c and d are positive constants, and so we have

$$C(\theta; \delta) = \theta^{c-1} \quad \text{and} \quad D(\theta; \delta) = \theta d, \tag{3.4}$$

where $\delta = (c, d)$.

3.1.1. *One-sample Bayesian Prediction.* The predictive survival function of $X_{s:n}$ in this special case is as follows:

Case I.

$$\begin{aligned} P(X_{s:n} > t \mid \underline{\mathbf{x}}) &= I_1^{-1} \sum_{w=0}^{s-r-1} \frac{C_1}{n-s+w+1} \int_0^\infty \theta^{r+c-1} \\ &\quad \times \exp\left[-\theta \left\{ \sum_{i=1}^r x_i + (n-r)x_r + (n-s+w+1)(t-x_r) + d \right\} \right] d\theta \\ &= I_1^{-1} \sum_{w=0}^{s-r-1} \frac{C_1}{n-s+w+1} \\ &\quad \times \left(\sum_{i=1}^r x_i + (n-r)x_r + (n-s+w+1)(t-x_r) + d \right)^{-(r+c)}, \end{aligned} \tag{3.5}$$

where

$$I_1 = \left(\sum_{i=1}^{r} x_i + (n-r)x_r + d \right)^{-(r+c)};$$

Case II.

$$P(X_{s:n} > t \mid \underline{\mathbf{x}}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_0^\infty \phi_k(T, \theta) g_w(t, T; \theta) \pi_2^*(\theta \mid \underline{\mathbf{x}}) d\theta, \tag{3.6}$$

where

$$\begin{aligned} \phi_k(T, \theta) &= \frac{\binom{n}{k} \exp[-(n-k)\theta T](1-\exp[-\theta T])^k}{\sum_{j=r}^{s-1} \binom{n}{j} \exp[-(n-j)\theta T](1-\exp[-\theta T])^j}, \\ g_w(t, T; \theta) &= \exp[-\theta\{(n-s+w+1)(t-T)\}], \\ \pi_2^*(\theta \mid \underline{\mathbf{x}}) &= \frac{d_1^{K+c}}{\Gamma(K+c)} \theta^{K+c-1} \exp(-\theta d_1), \end{aligned}$$

and $d_1 = \sum_{i=1}^{K} x_i + (n-K)T + d$.

It does not seem to be possible to compute the probability in (3.6) analytically. We therefore use Markov Chain Monte Carlo (MCMC) technique for constructing the Bayesian prediction interval.

To compute $\int_0^\infty f(\theta)\pi_2^*(\theta \mid \underline{\mathbf{x}})d\theta$ by using the MCMC technique, we use the following procedure:

Step 1. Generate $\theta_1 \sim \text{Gamma}(K+c, \sum_{i=1}^{K} x_i + (n-K)T + d)$.
Step 2. Repeat Step 1 and obtain $\theta_1, \theta_2, \ldots, \theta_N$.
Step 3. The approximate value of $\int_0^\infty f(\theta)\pi_2^*(\theta \mid \underline{\mathbf{x}})d\theta$ is then obtained as

$$\int_0^\infty f(\theta)\pi_2^*(\theta \mid \underline{\mathbf{x}})d\theta = \frac{\sum_{i=1}^{N} f(\theta_i)}{N}.$$

3.1.2. *Two-sample Bayesian Prediction.* The predictive survival function of $Y_{s:m}$ in this special case is as follows:

Case I.

$$\begin{aligned} P(Y_{s:m} > t \mid \underline{\mathbf{x}}) &= I_1^{-1} \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \int_0^\infty \theta^{r+c-1} \\ &\quad \times \exp\left[-\theta \left\{ \sum_{i=1}^{r} x_i + (n-r)x_r + (m-s+w+1)t + d \right\} \right] d\theta \\ &= I_1^{-1} \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \\ &\quad \times \left(\sum_{i=1}^{r} x_i + (n-r)x_r + (m-s+w+1)t + d \right)^{-(r+c)}, \end{aligned} \tag{3.7}$$

where

$$I_1 = \left(\sum_{i=1}^{r} x_i + (n-r)x_r + d \right)^{-(r+c)};$$

Case II.

$$\begin{aligned} P(Y_{s:m} > t \mid \underline{\mathbf{x}}) &= I_2^{-1} \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \int_0^\infty \theta^{K+c-1} \\ &\quad \times \exp\left[-\theta \left\{ \sum_{i=1}^{K} x_i + (n-K)T + (m-s+w+1)t + d \right\} \right] d\theta \\ &= I_2^{-1} \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \\ &\quad \times \left(\sum_{i=1}^{K} x_i + (n-K)T + (m-s+w+1)t + d \right)^{-(K+c)}, \end{aligned} \tag{3.8}$$

where

$$I_2 = \left(\sum_{i=1}^{K} x_i + (n-K)T + d \right)^{-(K+c)}.$$

3.2. *Pareto*(α, β) *Model*

The distribution function in this case is

$$\begin{aligned} F(x \mid \alpha, \beta) &= 1 - \left(\frac{\beta}{x} \right)^{\alpha} \\ &= 1 - \exp\left[-\alpha \ln\left(\frac{x}{\beta} \right) \right], \quad x > \beta, \end{aligned} \tag{3.9}$$

where $\alpha > 0$ and $\beta > 0$, and so we have

$$\lambda(x; \alpha, \beta) = \alpha \ln\left(\frac{x}{\beta} \right) \quad \text{and} \quad \lambda'(x; \alpha, \beta) = \frac{\alpha}{x}. \tag{3.10}$$

Under the assumption that both parameters α and β are unknown, we may consider a natural joint conjugate prior for α and β which was suggested by Lwin (1972) and generalized by Arnold and Press (1989). The generalized Lwin prior or the power-gamma prior, denoted by PG(a, b, c, d), is given by

$$\pi(\alpha, \beta; \delta) \propto \alpha^c \beta^{-1} \exp\left[-\alpha\left(d + a \ln\left(\frac{b}{\beta} \right) \right) \right], \quad \alpha > 0, \ 0 < \beta < b, \tag{3.11}$$

where a, b, c, d are positive constants. This general prior is obtained by first specifying the prior for the parameter α and then specifying the conditional prior for β, given knowledge on the parameter α. More specifically, we take $\pi(\alpha)$ as a gamma

distribution with parameters c and d, and $\pi(\beta \mid \alpha)$ as a power function distribution with parameters $a\alpha$ and b of the form

$$\pi(\beta \mid \alpha) \ \propto \alpha\beta^{a\alpha-1}b^{-a\alpha}, \quad 0 < \beta < b,$$

to arrive at the joint prior given in (3.11). Thus, we have

$$C(\alpha, \beta; \delta) = \alpha^c\beta^{-1} \quad \text{and} \quad D(\alpha, \beta; \delta) = \alpha\left(d + a\ln\left(\frac{b}{\beta}\right)\right), \tag{3.12}$$

where $\delta = (a, b, c, d)$.

3.2.1. *One-sample Bayesian Prediction.* The predictive survival function of $X_{s:n}$ in this special case is as follows:

Case I.

$$\begin{aligned}
&P(X_{s:n} > t \mid \underline{\mathbf{x}}) \\
&\quad = I_1^{-1}\sum_{w=0}^{s-r-1}\frac{C_1}{n-s+w+1}\int_0^L\int_0^\infty \alpha^{(r+c)}\beta^{-1}\left(\prod_{i=1}^r\frac{1}{x_i}\right)\exp\left[-\alpha\left\{\sum_{i=1}^r\ln\left(\frac{x_i}{\beta}\right)\right.\right. \\
&\qquad \left.\left. + (n-r)\ln\left(\frac{x_r}{\beta}\right) + a\ln\left(\frac{b}{\beta}\right) + (n-s+w+1)\ln\left(\frac{t}{x_r}\right) + d\right\}\right]d\alpha\, d\beta \\
&\quad = I_1^{-1}\sum_{w=0}^{s-r-1}\frac{C_1}{n-s+w+1}\left(\sum_{i=1}^r\ln\left(\frac{x_i}{L}\right) + (n-r)\ln\left(\frac{x_r}{L}\right) + a\ln\left(\frac{b}{L}\right)\right. \\
&\qquad \left. + (n-s+w+1)\ln\left(\frac{t}{x_r}\right) + d\right)^{-(r+c)},
\end{aligned} \tag{3.13}$$

where $L = \min(x_1, b)$ and

$$I_1 = \left(\sum_{i=1}^r\ln\left(\frac{x_i}{L}\right) + (n-r)\ln\left(\frac{x_r}{L}\right) + a\ln\left(\frac{b}{L}\right) + d\right)^{-(r+c)};$$

Case II:

$$\begin{aligned}
P(X_{s:n} > t \mid \underline{\mathbf{x}}) &= \sum_{k=r}^{s-1}\sum_{w=0}^{s-k-1}\frac{C_2}{n-s+w+1} \\
&\quad \times\int_0^L\int_0^\infty \phi_k(T; \alpha, \beta)g_w(t, T; \alpha, \beta)\pi_2^*(\alpha, \beta \mid \underline{\mathbf{x}})d\alpha\, d\beta,
\end{aligned} \tag{3.14}$$

where

$$\phi_k(T; \alpha, \beta) = \frac{\binom{n}{k}\exp\left[-(n-k)\alpha\ln\left(\frac{T}{\beta}\right)\right]\left(1-\exp\left[-\alpha\ln\left(\frac{T}{\beta}\right)\right]\right)^k}{\sum_{j=r}^{s-1}\binom{n}{j}\exp\left[-(n-j)\alpha\ln\left(\frac{T}{\beta}\right)\right]\left(1-\exp\left[-\alpha\ln\left(\frac{T}{\beta}\right)\right]\right)^j},$$

$$g_w(t, T; \alpha, \beta) = \exp\left[-(n-s+w+1)\alpha\ln\left(\frac{t}{T}\right)\right],$$

$$\pi_2^*(\alpha, \beta \mid \underline{\mathbf{x}}) = \pi_3^*(\alpha \mid \underline{\mathbf{x}})\pi_4^*(\beta \mid \alpha, \underline{\mathbf{x}}),$$
$$\pi_3^*(\alpha \mid \underline{\mathbf{x}}) = \frac{d_2^{K+c}}{\Gamma(K+c)}\alpha^{K+c-1}\exp(-\alpha d_2),$$
$$\pi_4^*(\beta \mid \alpha, \underline{\mathbf{x}}) = \alpha(n+a)\beta^{\alpha(n+a)-1}L^{-\alpha(n+a)},$$

and $d_2 = \sum_{i=1}^{K}\ln(\frac{x_i}{L}) + (n-K)\ln(\frac{T}{L}) + a\ln(\frac{b}{L}) + d$.

It does not seem to be possible to compute the probability in (3.14) analytically. We therefore use the Gibbs sampling technique to generate MCMC samples, and then use the MCMC technique for constructing the Bayesian prediction interval.

To compute $\int_0^L\int_0^\infty g(\alpha, \beta)\pi_2^*(\alpha, \beta \mid \underline{\mathbf{x}})d\alpha\, d\beta$ by using the MCMC technique, we use the following procedure:

Step 1. Generate $\alpha_1 \sim$ Gamma$(K+c, \sum_{i=1}^{K}\ln\left(\frac{x_i}{L}\right) + (n-K)\ln\left(\frac{T}{L}\right) + a\ln\left(\frac{b}{L}\right) + d)$;
Step 2. Generate $\beta_1 \sim$ Power function$(\alpha_1(n+a), L)$;
Step 3. Repeat Steps 1 and 2 and obtain $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_N, \beta_N)$;
Step 4. The approximate value of $\int_0^L\int_0^\infty g(\alpha, \beta)\pi_2^*(\alpha, \beta \mid \underline{\mathbf{x}})d\alpha\, d\beta$ is then obtained as

$$\int_0^L\int_0^\infty g(\alpha, \beta)\pi_2^*(\alpha, \beta \mid \underline{\mathbf{x}})d\alpha\, d\beta = \frac{\sum_{i=1}^{N} g(\alpha_i, \beta_i)}{N}.$$

3.2.2. *Two-sample Bayesian Prediction.* The predictive survival function of $Y_{s:m}$ in this special case is as follows:

Case I.

$$\begin{aligned} P(Y_{s:m} > t \mid \underline{\mathbf{x}}) &= I_1^{-1}\sum_{w=0}^{s-1}\frac{C_3}{m-s+w+1}\int_0^L\int_0^\infty \alpha^{r+c}\left(\prod_{i=1}^{r}\frac{1}{x_i}\right)\exp\left[-\alpha\left\{\sum_{i=1}^{r}\ln\left(\frac{x_i}{\beta}\right)\right.\right. \\ &\quad \left.\left. + (n-r)\ln\left(\frac{x_r}{\beta}\right) + a\ln\frac{b}{\beta} + (m-s+w+1)\ln\left(\frac{t}{\beta}\right) + d\right\}\right]d\alpha\, d\beta \\ &= I_1^{-1}\sum_{w=0}^{s-1}\frac{C_3}{(m-s+w+1)(n+a+m-s+w+1)} \\ &\quad \times\left(\sum_{i=1}^{r}\ln\left(\frac{x_i}{L}\right) + (n-r)\ln\left(\frac{x_r}{L}\right) + a\ln\frac{b}{L}\right. \\ &\quad \left. + (m-s+w+1)\ln\left(\frac{t}{L}\right) + d\right)^{-(r+c)}, \end{aligned} \tag{3.15}$$

where

$$I_1 = \frac{1}{n+a}\left(\sum_{i=1}^{r}\ln\left(\frac{x_i}{L}\right) + (n-r)\ln\left(\frac{x_r}{L}\right) + a\ln\frac{b}{L} + d\right)^{-(r+c)};$$

Case II.

$$P(Y_{s:m} > t \mid \underline{\mathbf{x}}) = I_2^{-1}\sum_{w=0}^{s-1}\frac{C_3}{m-s+w+1}\int_0^L\int_0^\infty \alpha^{K+c}\left(\prod_{i=1}^{K}\frac{1}{x_i}\right)\exp\left[-\alpha\left\{\sum_{i=1}^{K}\ln\left(\frac{x_i}{\beta}\right)\right.\right.$$

$$+ (n-K)\ln\left(\frac{T}{\beta}\right) + a\ln\frac{b}{\beta} + (m-s+w+1)\ln\left(\frac{t}{\beta}\right) + d\bigg\}\bigg]d\alpha$$

$$= I_2^{-1}\sum_{w=0}^{s-1}\frac{C_3}{(m-s+w+1)(n+a+m-s+w+1)}$$

$$\times\left(\sum_{i=1}^{K}\ln\left(\frac{x_i}{L}\right) + (n-K)\ln\left(\frac{T}{L}\right) + a\ln\frac{b}{L}\right.$$

$$\left. + (m-s+w+1)\ln\left(\frac{t}{L}\right) + d\right)^{-(K+c)}, \tag{3.16}$$

where

$$I_2 = \frac{1}{n+a}\left(\sum_{i=1}^{K}\ln\left(\frac{x_i}{L}\right) + (n-K)\ln\left(\frac{T}{L}\right) + a\ln\frac{b}{L} + d\right)^{-(K+c)}.$$

4. Numerical Results

To illustrate the inferential procedures developed in the preceding sections, we present a numerical study for two distributions. The first distribution is the exponential(θ) distribution when θ is unknown, while the second one is the Pareto(α,β) distribution when both parameters α and β are unknown.

Example 4.1. To illustrate the prediction results for the exponential(θ) distribution, when θ is unknown, let us consider the data given by Bartholomew (1963) consisting of 20 items placed on a life test for a pre-fixed time of 150 h. During that period, 15 items failed with the following lifetimes, measured in hours: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, and 138.

We shall use these data to consider two different Type-II hybrid censoring schemes:

1. When $r = 13$ and $T = 100$. Since $T > x_{13:20}$, the testing would have terminated in this case at T and we would have obtained the following data: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, and 99.
2. When $r = 15$ and $T = 100$. Since $x_{15:20} > T$, the testing would have terminated in this case at time $x_{15:20} = 138$ and we would have obtained the following data: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, and 138.

As done previously by Bartholomew (1963) and Childs et al. (2003), we assume these data to have come from the exponential(θ) distribution, where θ is unknown. Based on the above two Type-II hybrid censoring schemes, we then used the results presented earlier in sec. 3.1 to construct 95% one-sample Bayesian prediction intervals for future order statistics $X_{s:n}$, $s = 16, \ldots, 20$, from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:m}$, $s = 1, \ldots, 20$, from a future sample of size $m = 20$. To examine the sensitivity of the Bayesian prediction intervals with respect to the hyperparameters (c, d), Table 1 presents the lower and upper 95% one-sample Bayesian prediction bounds for $X_{s:n}$, $s = 16, \ldots, 20$, for the choices of $c = 0.9$, 1, 1.1 and $d = 50$, 55, 60. The lower and

Table 1
95% one-sample Bayesian prediction bounds for $X_{s:n}$, $s = 16, \ldots, 20$, from the exponential distribution

		$r = 13$ and $T = 10$						$r = 15$ and $T = 100$					
		$d = 50$		$d = 55$		$d = 60$		$d = 50$		$d = 55$		$d = 60$	
c	s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
09	16	101.584	230.341	101.599	230.888	101.615	231.436	138.503	220.359	138.504	220.620	138.506	220.881
	17	103.193	283.859	103.231	284.627	103.270	285.394	143.252	282.417	143.269	282.875	143.285	283.333
	18	107.335	357.982	107.437	359.038	107.540	360.094	153.096	361.674	153.144	362.383	153.192	363.092
	19	117.996	475.301	118.232	476.802	118.469	478.303	169.317	483.830	169.417	484.926	169.516	486.023
	20	145.245	739.427	145.670	741.923	146.096	744.419	198.546	756.425	198.738	758.385	198.930	760.346
1	16	101.558	229.123	101.572	229.667	101.588	230.209	138.499	219.783	138.501	220.042	138.503	220.301
	17	103.125	282.131	103.164	282.895	103.202	283.657	143.220	281.384	143.237	281.838	143.253	282.293
	18	107.163	355.593	107.263	356.641	107.364	357.689	153.006	360.051	153.053	360.755	153.101	361.459
	19	117.604	471.901	117.836	473.391	118.070	474.879	169.131	481.304	169.230	482.392	169.329	483.480
	20	144.538	733.824	144.961	736.301	145.384	738.776	198.190	751.921	198.380	753.868	198.571	755.814
1.1	16	101.532	227.881	101.546	228.420	101.560	228.959	138.496	219.214	138.498	219.471	138.499	219.729
	17	103.059	280.371	103.097	281.127	103.134	281.883	143.188	280.365	143.205	280.816	143.221	281.268
	18	106.994	353.157	107.092	354.197	107.190	355.236	152.916	358.452	152.963	359.151	153.011	359.850
	19	117.218	468.435	117.447	469.916	117.677	471.391	168.948	478.813	169.046	479.894	169.144	480.975
	20	143.837	728.117	144.258	730.573	144.678	733.029	197.837	747.483	198.027	749.415	198.217	751.348

Table 2

95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1, \ldots, 20$, from the exponential distribution

	$r = 13$ and $T = 100$						$r = 15$ and $T = 100$					
(c, d)	(1, 50)		(1, 55)		(1, 60)		(1, 50)		(1, 55)		(1, 60)	
s	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$
1	0.121	20.198	0.122	20.274	0.122	20.349	0.125	20.446	0.125	20.510	0.126	20.575
2	1.159	32.290	1.163	32.410	1.167	32.531	1.197	32.564	1.201	32.668	1.204	32.771
3	2.978	43.974	2.989	44.138	3.000	44.302	3.084	44.237	3.094	44.378	3.103	44.518
4	5.306	55.874	5.326	56.083	5.345	56.291	5.504	56.103	5.522	56.281	5.539	56.459
5	8.027	68.257	8.055	68.512	8.084	68.766	8.337	68.433	8.364	68.650	8.390	68.867
6	11.087	81.307	11.128	81.611	11.170	81.914	11.533	81.416	11.570	81.674	11.606	81.932
7	14.480	95.192	14.534	95.547	14.588	95.903	15.079	95.221	15.127	95.523	15.175	95.824
8	18.213	110.091	18.281	110.501	18.349	110.912	18.983	110.026	19.044	110.375	19.104	110.724
9	22.311	126.210	22.394	126.681	22.477	127.152	23.272	126.041	23.346	126.440	23.420	126.840
10	26.813	143.806	26.913	144.342	27.013	144.879	27.988	143.519	28.077	143.974	28.166	144.429
11	31.780	163.206	31.898	163.815	32.017	164.424	33.193	162.791	33.298	163.307	33.403	163.823
12	37.292	184.848	37.431	185.537	37.570	186.227	38.970	184.292	39.094	184.876	39.217	185.460
13	43.462	209.337	43.624	210.118	43.786	210.900	45.439	208.628	45.583	209.289	45.727	209.951
14	50.451	237.552	50.639	238.438	50.827	239.324	52.765	236.678	52.932	237.428	53.100	238.179
15	58.491	270.831	58.709	271.842	58.927	272.852	61.194	269.783	61.388	270.639	61.582	271.494
16	67.944	311.376	68.198	312.538	68.451	313.700	71.100	310.151	71.326	311.134	71.551	312.117
17	79.408	363.170	79.704	364.525	80.000	365.880	83.108	361.781	83.372	362.928	83.635	364.075
18	93.988	434.569	94.338	436.191	94.689	437.812	98.368	433.086	98.680	434.460	98.992	435.833
19	114.130	548.018	114.556	550.063	114.982	552.108	119.419	546.721	119.798	548.454	120.176	550.188
20	147.597	805.243	148.147	808.247	148.698	811.252	154.298	805.735	154.788	808.290	155.277	810.845

upper 95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1, \ldots, 20$, for the choices of $c = 1$ and $d = 50$, 55, 60, are presented in Table 2.

Example 4.2. To illustrate the prediction results for the Pareto(α, β) distribution, when both parameters α and β are unknown, we generated order statistics from a sample of size $n = 20$ from the Pareto distribution. The generated order statistics from the Pareto distribution (with $\alpha = 3$ and $\beta = 6$) are as follows: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, 9.290, 9.360, 9.525, 9.836, 10.263, 11.113, 15.769, and 39.211.

We shall use these data to consider two different Type-II hybrid censoring schemes.

1. When $r = 13$ and $T = 9.3$. Since $T > x_{13:20}$, the testing would have terminated in this case at T and we would have obtained the following data: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, and 9.290.
2. When $r = 15$ and $T = 9.3$. Since $x_{15:20} > T$, the testing would have terminated in this case at time $x_{15:20} = 9.525$ and we would have obtained the following data: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, 9.290, 9.360, and 9.525.

We assume these data to have come from the Pareto(α, β) distribution, where both parameters α and β are unknown. Based on the above two Type-II hybrid censoring schemes, we then used the results presented earlier in sec. 3.2 to construct 95% one-sample Bayesian prediction intervals for future order statistics $X_{s:n}$, $s = 16, \ldots, 20$, from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:m}$, $s = 1, \ldots, 20$, from a future sample of size $m = 20$. To examine the sensitivity of the Bayesian prediction intervals with respect to the hyperparameters (a, b, c, d), we used three different choices of the hyperparameters (a, b, c, d): $(1, 9, 3, 1)$, $(1, 9, 6, 2)$, $(1, 9, 9, 3)$. The corresponding results for one-sample and two-sample predictions, for the three choices of the hyperparameters are presented in Tables 3 and 4, respectively.

Remarks.

1. From Tables 1–4, we notice that, when we use the same value of T but larger r, the Bayesian prediction bounds become tighter as expected since the duration of the life-testing experiment is longer in this case.
2. It is evident from Tables 1 and 2 that, in the case of the exponential distribution, the lower as well as upper bounds are relatively insensitive to the specification of the hyperparameters (c, d).
3. It is evident from Tables 3 and 4 that, in the case of the Pareto distribution, the lower bounds are relatively insensitive to the specification of the hyperparameters (a, b, c, d) while the upper bounds are somewhat sensitive.
4. If the vector of prior parameters δ is unknown, the empirical Bayes approach could be used in estimating such prior parameters based on past samples; see, for example, Maritz and Lwin (1989). Alternatively, one could use the hierarchical Bayesian method in which some suitable prior for δ could be proposed; see, for example, Geisser (1990) and Bernardo and Smith (1994). Work in these directions are currently under progress and we hope to report these findings in a future article.

Table 3
95% two-sample Bayesian prediction bounds for $X_{s:n}$, $s = 16, \ldots, 20$, from the Pareto distribution

	$r = 13$ and $T = 9.3$						$r = 15$ and $T = 9.3$					
(a, b, c, d)	(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)		(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
16	9.362	15.895	9.389	17.589	9.339	15.284	9.543	12.914	9.546	13.515	9.542	12.688
17	9.427	19.765	9.490	22.590	9.401	18.702	9.715	16.217	9.744	17.559	9.705	15.715
18	9.593	26.794	9.755	32.215	9.520	24.829	10.082	21.683	10.168	24.520	10.053	20.644
19	10.050	43.510	10.421	56.546	9.861	39.046	10.719	33.938	10.910	41.035	10.656	31.443
20	11.336	130.715	12.055	202.019	10.923	109.213	11.970	92.487	12.386	129.907	11.834	80.712

Table 4
95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1, \ldots, 20$, from the Pareto distribution

	$r = 13$ and $T = 9.3$						$r = 15$ and $T = 9.3$					
(a, b, c, d)	(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)		(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)	
s	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$
1	6.048	6.546	6.049	6.631	6.048	6.512	6.048	6.494	6.049	6.568	6.048	6.467
2	6.054	6.796	6.056	6.927	6.054	6.744	6.054	6.716	6.055	6.829	6.053	6.674
3	6.066	7.075	6.070	7.259	6.065	7.001	6.065	6.961	6.068	7.118	6.064	6.903
4	6.090	7.386	6.097	7.631	6.087	7.286	6.086	7.232	6.092	7.441	6.084	7.155
5	6.131	7.733	6.145	8.050	6.126	7.604	6.123	7.534	6.136	7.802	6.119	7.434
6	6.192	8.123	6.217	8.523	6.184	7.959	6.180	7.870	6.202	8.207	6.173	7.745
7	6.274	8.562	6.312	9.062	6.261	8.358	6.254	8.247	6.288	8.664	6.244	8.092
8	6.372	9.063	6.426	9.682	6.353	8.810	6.344	8.674	6.393	9.186	6.329	8.483
9	6.483	9.640	6.558	10.402	6.458	9.329	6.446	9.161	6.512	9.787	6.426	8.928
10	6.608	10.313	6.705	11.251	6.575	9.931	6.562	9.726	6.648	10.489	6.536	9.442
11	6.748	11.111	6.871	12.270	6.708	10.642	6.689	6.689	6.799	11.323	6.657	10.044
12	6.906	12.075	7.059	13.517	6.857	11.496	6.835	11.185	6.970	12.334	6.794	10.762
13	7.086	13.269	7.272	15.084	7.027	12.546	6.999	12.160	7.164	13.588	6.950	11.638
14	7.296	14.794	7.523	17.118	7.224	13.878	7.191	13.392	7.391	15.196	7.131	12.738
15	7.542	16.823	7.820	19.877	7.455	15.635	7.415	15.009	7.659	17.342	7.343	14.173
16	7.842	19.679	8.182	23.853	7.735	18.084	7.687	17.250	7.986	20.378	7.600	16.147
17	8.220	24.052	8.643	30.123	8.089	21.788	8.029	20.618	8.399	25.056	7.921	19.085
18	8.726	31.738	9.264	41.587	8.559	28.193	8.484	26.389	8.953	33.353	8.349	24.055
19	9.473	49.386	10.192	69.547	9.252	42.541	9.152	39.153	9.775	52.686	8.974	34.839
20	10.851	135.358	11.936	224.665	10.520	108.940	10.371	96.642	11.300	150.128	10.107	81.499

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