One- and Two-Sample Bayesian Prediction Intervals Based on Type-II Hybrid Censored Data

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One- and Two-Sample Bayesian Prediction Intervals Based on Type-II Hybrid Censored Data

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In this article, one- and two-sample Bayesian prediction intervals based on Type-II hybrid censored data are derived. For the illustration of the developed results, the Exponential(\(\theta\)) and Pareto(\(\alpha, \beta\)) distributions are used as examples. One-sample Bayesian predictive survival function cannot be obtained in closed form. Gibbs sampling procedure is therefore used to draw Markov Chain Monte Carlo (MCMC) samples, and they are in turn used to compute the approximate predictive survival function, and the corresponding numerical results are presented.

Keywords Bayesian prediction; Exponential distribution; Markov Chain Monte Carlo; Order statistics; Pareto distribution; Type-II hybrid censored sample.

Mathematics Subject Classification Primary 62N01; Secondary 62F15.

1. Introduction

In reliability analysis, experiments often terminate before all units on test have failed due to cost and time considerations. In such cases, failure information is available only on part of the sample, and on all units that had not failed, one has only partial information. Such data are said to be censored data. The two most common censoring schemes are Type-I and Type-II censoring schemes. They can be described as follows: Consider \(n\) identical units under observation in a life-testing experiment. In the Type-I censoring scheme, the experiment is terminated when a pre-fixed censoring time \(T\) arrives. On the other hand, in the Type-II censoring scheme, the experiment gets terminated when a pre-specified number \(r \leq n\) of failures occur. In both censoring schemes, some information is lost as only a part of the sample is observed, but they do result in a saving in terms of time and cost. In a Type-I censoring scheme, the termination time is guaranteed but the level of efficiency may be too low or too high due to the uncertainty in the number of complete failures. In a Type-II censoring scheme, the level of efficiency is guaranteed (since the number of
failures to be observed is fixed in advance), but the duration of the experiment may end up being too long since the exact time of the $r$th failure is uncertain. For these reasons, another censoring scheme becomes necessary if both efficiency level and guaranteed termination time are sought. A mixture of Type-I and Type-II censoring schemes, known as hybrid censoring scheme, has been discussed in the literature for this purpose.

Epstein (1954) introduced the Type-I hybrid censoring scheme (Type-I HCS) in which the life-testing experiment is terminated as soon as a pre-specified number $r$ out of $n$ items has failed or a pre-fixed time $T$ has been reached in the life-test. The Type-I HCS has been used as a reliability acceptance test in MIL-STD-781 C (1977). However, the Type-I HCS may result in too few observations in the data. For this reason, Childs et al. (2003) proposed the Type-II hybrid censoring scheme (Type-II HCS) in which the life-testing experiment gets terminated whenever the later of the two stopping rules is reached. The Type-II HCS is preferable to use as it gives a guarantee that the number of observations in the data is at least $r$, thus resulting in more efficient inferential procedures than under Type-I HCS. A hybrid censoring has been discussed extensively in the reliability literature; see, for example, Epstein (1954), Fairbanks et al. (1982), Chen and Bhattacharyya (1988), Jeong et al. (1996), Gupta and Kundu (1998), Childs et al. (2003), Chandrasekar et al. (2004), Kundu (2007), Lin et al. (2008), Kundu and Banerjee (2008), Park et al. (2008), and Park and Balakrishnan (2009).

In many practical problems, one would wish to use past data for predicting a future observation from the same population. One way to do this is to construct an interval that will contain the future observation with a specified probability, called as a prediction interval. Bayesian prediction bounds for future observations based on Type-I censored data have been discussed by several authors, including Evans and Nigm (1980), AL-Hussaini (1999a), and AL-Hussaini et al. (2001). Several authors have studied the Bayesian prediction based on Type-II censored data; see Dunsmore (1974), Nigm and Hamdy (1987), Nigm (1988, 1989), AL-Hussaini and Jaheen (1995), AL-Hussaini (1999b), and Raqab and Madi (2005). Gupta and Kundu (1998) and Kundu (2007) derived Bayes estimates and the corresponding highest posterior density credible intervals for the unknown parameters of exponential and Weibull distributions based on Type-I hybrid censored data. Draper and Guttman (1987) discussed the two-sample Bayesian prediction of a future observation from an exponential distribution based on Type-I hybrid censored data. Ebrahimi (1992) developed the classical prediction intervals for future failures in the case of exponential distribution under Type-I hybrid censoring.

In this article, we consider a general form for the underlying distribution and a general conjugate prior, and describe a general procedure for determining the Bayesian prediction intervals for future life-lengths based on observed Type-II hybrid censored data. In Sec. 2, we derive the one-sample Bayesian predictive survival function and the one-sample Bayesian prediction bounds for the $s$th ($r < s \leq n$) observation from a Type-II hybrid censored sample. Next, we derive the two-sample Bayesian predictive survival function and the two-sample Bayesian prediction bounds for the $s$th observation from a future independent sample when the (observed) informative sample is Type-II hybrid censored and the (unobserved) future sample is the usual order statistics from the same parent distribution. In Sec. 3, we present the results for the exponential ($\theta$) and Pareto($\alpha, \beta$) distributions...
as illustrative examples and we use Markov Chain Monte Carlo method to compute the approximate predictive survival function in the one-sample case. Finally, in Sec. 4, we present some numerical results for these examples.

Let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$ be the order statistics from a random sample of size $n$ from an absolutely continuous distribution function $F(x) \equiv F(x \mid \theta)$ with density function $f(x) \equiv f(x \mid \theta)$, where the parameter $\theta \in \Theta$ may be a real vector. Let $K$ denote the number of $X_{i,n}$’s that are at most $T$. Therefore, under the Type-II hybrid censoring scheme described above, we have one of the two following types of observations:

**Case I.** $X_{1,n} < \cdots < X_{r,n}$ if $X_{r,n} > T$ with $0 \leq K \leq r - 1$;

**Case II.** $X_{1,n} < \cdots < X_{K,n}$ if $T \geq X_{r,n}$ with $r \leq K \leq n$.

Thus, the likelihood function of the Type-II hybrid censored sample is as follows:

**Case I.**

$$f(\mathbf{x}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i)(1 - F(x_r))^{n-r}, \quad (1.1)$$

where $\mathbf{x} = (x_1, \ldots, x_r)$, $x_1 < \cdots < x_r$ and $x_r > T$;

**Case II.**

$$f(\mathbf{x}) = \frac{n!}{(n-K)!} \prod_{i=1}^{K} f(x_i)(1 - F(T))^{n-K}, \quad (1.2)$$

where $\mathbf{x} = (x_1, \ldots, x_K)$ and $x_1 < \cdots < x_K \leq T < x_{K+1}$.

**Lemma 1.1.** Conditional on $K = k$, the vectors $(X_{1,n}, \ldots, X_{k,n})$ and $(X_{k+1,n}, \ldots, X_{n,n})$ are mutually independent with

$$
\begin{align*}
(X_{1,n}, \ldots, X_{k,n}) &\overset{d}{=} (V_{1,k}, \ldots, V_{k,k}), \\
(X_{k+1,n}, \ldots, X_{n,n}) &\overset{d}{=} (W_{1,n-k}, \ldots, W_{n-k,n-k}),
\end{align*}
$$

where $V_{1,k}, \ldots, V_{k,k}$ are order statistics from an iid sample of size $k$ from $F(x)$ right-truncated at $T$, and $W_{1,n-k}, \ldots, W_{n-k,n-k}$ are order statistics from an iid sample of size $n-k$ from $F(x)$ left-truncated at $T$.

For a proof of this result as well as some generalizations of this result, one may refer to Iliopoulos and Balakrishnan (2009).

Let $r < s \leq n$, then the conditional density function of $X_{s,n}$, given the Type-II hybrid censored sample, is as follows:

**Case I.**

$$f(x_s \mid \mathbf{x}) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{(F(x_s) - F(x_r))(s-r-1)(1 - F(x_r))^{n-s}f(x_s)}{(1 - F(x_r))^{n-r}}, \quad (1.3)$$

where $\mathbf{x} = (x_1, \ldots, x_r)$ and $x_s > x_r$;
We consider here a general conjugate prior, suggested by AL-Hussaini (1999b), that for the Bayesian prediction setup, we need a suitable prior parameter distribution.

2. Bayesian Prediction Intervals

Case II.

\[ f(x_s | \mathbf{x}) = \frac{1}{P(r \leq K \leq s - 1)} \sum_{k=r}^{s-1} f(x_s | \mathbf{x}, K = k) P(K = k) \]

\[ = \frac{1}{(s - k - 1)!} \frac{(n - k)! \phi_k(T)}{(1 - F(T))^{n-k}} \frac{1}{(F(x_s) - F(T))^{s-1}} (1 - F(x_s))^{n-s} f(x_s), \quad (1.4) \]

where \( \mathbf{x} = (x_1, \ldots, x_k), x_s > T \) and \( \phi_k(T) = \frac{P(K = k)}{\sum_{j=r}^{n} P(K = j)} \).

Let \( Y_1 \leq Y_2 \leq \cdots \leq Y_m \) be the order statistics (OS) from a future random sample of size \( m \) from the same population. Then the marginal density function of the \( s \)th OS \( Y_{s,m} \) is given by

\[ f_{Y_{s,m}}(y_s) = \frac{m!}{(s - 1)! (m - s)!} (F(y_s))^{s-1} (1 - F(y_s))^{m-s} f(y_s), \quad (1.5) \]

where \( 1 \leq s \leq m \).

Since the survival function (SF) \( \overline{F}(x | \theta) = 1 - F(x | \theta) \) corresponding to any cumulative distribution function (cdf) \( F(x | \theta), \theta \in \Theta \), can be written in the form

\[ \overline{F}(x | \theta) = \exp[-\lambda(x; \theta)], \quad (1.6) \]

where \( \lambda(x; \theta) \) is continuous, monotone increasing and differentiable function, with \( \lambda(x; \theta) \to 0 \) as \( x \to -\infty \) and \( \lambda(x; \theta) \to \infty \) as \( x \to \infty \). The probability density function (pdf) corresponding to (1.6) is given by

\[ f(x | \theta) = \lambda'(x; \theta) \exp[-\lambda(x; \theta)], \quad (1.7) \]

where \( \lambda'(x; \theta) \) is the derivative of \( \lambda(x; \theta) \) with respect to \( x \).

With an appropriate choice of \( \lambda(x; \theta) \) (notice that the derivative of \( \lambda(x; \theta) \) with respect to \( x \) is the hazard rate function), several distributions that are used in reliability studies can be obtained as special cases. For example, if \( \lambda(x; \theta) = \theta x \), we obtain the exponential(\( \theta \)) distribution. If \( \lambda(x; \theta) = -a \ln(x) \), we obtain the Pareto(\( a, \beta \)) distribution. If \( \lambda(x; \theta) = x^\beta \), we obtain the Weibull(\( a, \beta \)) distribution. The Burr Type XII(\( a, \beta \)) distribution is obtained by taking \( \lambda(x; \theta) = a \ln(1 + x^\beta) \).

Appropriate conditions need to be imposed on \( \lambda(x; \theta) \) to suit the domain on which \( \overline{F}(x | \theta) \) in (1.6) is defined. For example, if \( \overline{F}(x | \theta) \) is defined only on the positive half of the real line (as for the exponential, Weibull, and Burr Type XII distributions), then \( \lambda(x; \theta) \to 0 \) as \( x \to 0^+ \) and \( \lambda(x; \theta) \to \infty \) as \( x \to \infty \). If \( \overline{F}(x | \theta) \) is defined on \( (\beta, \infty) \) (as in the Pareto distribution), then \( \lambda(x; \theta) \to 0 \) as \( x \to \beta^+ \) and \( \lambda(x; \theta) \to \infty \) as \( x \to \infty \). The exponential form of the SF in (1.6) provides some flexibility in developing general results as done in the following sections.

2. Bayesian Prediction Intervals

For the Bayesian prediction setup, we need a suitable prior parameter distribution. We consider here a general conjugate prior, suggested by AL-Hussaini (1999b), that
is given by

$$\pi(\theta; \delta) \propto C(\theta; \delta) \exp[-D(\theta; \delta)],$$  \hspace{1cm} (2.1)

where $\theta \in \Theta$ is the vector of parameters of the distribution under consideration and $\delta$ is the vector of prior parameters. The prior family in (2.1) includes several priors used in the literature as special cases.

### 2.1. One-sample Bayesian prediction

Upon using (1.6) and (1.7) in (1.1) and (1.2), we obtain the likelihood function as follows:

**Case I.**

$$L(\theta; \underline{x}) = \frac{n!}{(n-r)!} \left( \prod_{i=1}^{r} \tilde{\lambda}(x_i, \theta) \right) \exp \left[ - \left\{ \sum_{i=1}^{r} \tilde{\lambda}(x_i; \theta) + (n-r)\lambda(x_r; \theta) \right\} \right],$$

(2.2)

where $\underline{x} = (x_1, \ldots, x_r), x_1 < \cdots < x_r$ and $x_r > T$;

**Case II.**

$$L(\theta; \underline{x}) = \frac{n!}{(n-K)!} \left( \prod_{i=1}^{K} \tilde{\lambda}(x_i, \theta) \right) \exp \left[ - \left\{ \sum_{i=1}^{K} \tilde{\lambda}(x_i; \theta) + (n-K)\lambda(T; \theta) \right\} \right],$$

(2.3)

where $\underline{x} = (x_1, \ldots, x_K)$ and $x_1 < \cdots < x_K \leq T < x_{K+1}$.

Similarly, upon substituting (1.6) and (1.7) in (1.3) and (1.4), we obtain the conditional density function of $X_{s:n}$, given the Type-II hybrid censored sample, as follows:

**Case I.**

$$f(x_s | \underline{x}) = \sum_{w=0}^{s-r-1} C_1 \tilde{\lambda}(x_s; \theta) g_w(x_s, x_r; \theta),$$

(2.4)

where $\underline{x} = (x_1, \ldots, x_r), x_s > x_r$, $C_1 = \frac{(-1)^s \lambda(x_s; \theta)}{(s-r-1)!} (n-r)!$ and

$$g_w(x, y; \theta) = \exp[-(n-s+w+1)(\lambda(x; \theta) - \lambda(y; \theta))];$$

(2.5)

**Case II.**

$$f(x_s | \underline{x}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} C_2 \phi_k(T, \theta) \tilde{\lambda}(x_s; \theta) g_w(x_s, T; \theta),$$

(2.6)
where $\mathbf{x} = (x_1, \ldots, x_K), x_i > T, C_2 = \frac{(-1)^s (s - k - 1)^{(n - k)!}}{(n - k - 1)!}$. and

$$
\phi_k(T, \theta) = \frac{\left(\frac{n}{s}ight) \exp[-{(n - k)}\lambda(T; \theta)](1 - \exp[-\lambda(T; \theta)])^k}{\sum_{j=1}^{s} \left(\frac{n}{s}ight) \exp[-{(n - j)}\lambda(T; \theta)](1 - \exp[-\lambda(T; \theta)])^j}.
$$

(2.7)

Since the posterior density function is given by

$$
\pi^*(\theta \mid \mathbf{x}) = I^{-1} \pi(\theta; \delta)L(\theta; \mathbf{x}),
$$

(2.8)

where

$$
I = \int_{\theta \in \Theta} \pi(\theta; \delta)L(\theta; \mathbf{x})d\theta,
$$

(2.9)

we obtain from (2.1), (2.2), (2.3), and (2.8) the posterior density function as follows:

**Case I.**

$$
\pi_1^*(\theta \mid \mathbf{x}) = I_1^{-1} \eta_1(\theta; \mathbf{x}) \exp[-\zeta_1(\theta; \mathbf{x})],
$$

(2.10)

where

$$
\eta_1(\theta; \mathbf{x}) = C(\theta; \delta) \prod_{i=1}^{r} \lambda'(x_i; \theta),
$$

$$
\zeta_1(\theta; \mathbf{x}) = \sum_{i=1}^{r} \lambda(x_i; \theta) + (n - r)\lambda(T; \theta) + D(\theta; \delta),
$$

and

$$
I_1 = \int_{\theta \in \Theta} \eta_1(\theta; \mathbf{x}) \exp[-\zeta_1(\theta; \mathbf{x})]d\theta;
$$

**Case II.**

$$
\pi_2^*(\theta \mid \mathbf{x}) = I_2^{-1} \eta_2(\theta; \mathbf{x}) \exp[-\zeta_2(\theta; \mathbf{x})],
$$

(2.11)

where

$$
\eta_2(\theta; \mathbf{x}) = C(\theta; \delta) \prod_{i=1}^{K} \lambda'(x_i; \theta),
$$

$$
\zeta_2(\theta; \mathbf{x}) = \sum_{i=1}^{K} \lambda(x_i; \theta) + (n - K)\lambda(T; \theta) + D(\theta; \delta),
$$

and

$$
I_2 = \int_{\theta \in \Theta} \eta_2(\theta; \mathbf{x}) \exp[-\zeta_2(\theta; \mathbf{x})]d\theta.$$
Bayesian Prediction based on Hybrid Censoring

Since the Bayesian predictive density function of \(X_{s,n}\) is given by

\[
p(x_s | \bar{x}) = \int \frac{L}{\theta} \in \Theta \ f(x_s | \bar{x}) \pi^*(\theta | \bar{x}) \, d\theta,
\]

upon using (2.4), (2.6), (2.10), and (2.11) in (2.12), we obtain the predictive density function of \(X_{s,n}\) as follows:

**Case I.**

\[
p(x_s | \bar{x}) = \sum_{u=0}^{s-r-1} C_1 \int \frac{L}{\theta} \in \Theta \ \lambda'(x_s, \theta) g_w(x_s, x_r; \theta) \pi^*_1(\theta | \bar{x}) \, d\theta,
\]

where \(\bar{x} = (x_1, \ldots, x_r)\) and \(x_s > x_r\);

**Case II.**

\[
p(x_s | \bar{x}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} C_2 \int \frac{L}{\theta} \in \Theta \ \lambda'(x_s, \theta) \phi_k(T, \theta) g_w(x_s, T; \theta) \pi^*_2(\theta | \bar{x}) \, d\theta,
\]

where \(\bar{x} = (x_1, \ldots, x_k)\) and \(x_s > T\).

So, the predictive survival function of \(X_{s,n}\) is obtained as follows:

**Case I.**

\[
P(X_{s,n} > t | \bar{x}) = \sum_{u=0}^{s-r-1} C_1 \int_t^\infty \int \frac{L}{\theta} \in \Theta \ \lambda'(x_s, \theta) g_w(x_s, x_r; \theta) \pi^*_1(\theta | \bar{x}) \, d\theta \, dx_s
\]

\[
= \sum_{u=0}^{s-r-1} \frac{C_1}{n - s + w + 1} \int \frac{L}{\theta} \in \Theta \ g_w(t, x_r; \theta) \pi^*_1(\theta | \bar{x}) \, d\theta; \quad (2.15)
\]

**Case II.**

\[
P(X_{s,n} > t | \bar{x}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} C_2 \int_t^\infty \int \frac{L}{\theta} \in \Theta \ \lambda'(x_s, \theta) \phi_k(T, \theta) g_w(x_s, T; \theta) \pi^*_2(\theta | \bar{x}) \, d\theta \, dx_s
\]

\[
= \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n - s + w + 1} \int \frac{L}{\theta} \in \Theta \ \phi_k(T, \theta) g_w(t, T; \theta) \pi^*_2(\theta | \bar{x}) \, d\theta. \quad (2.16)
\]

Then, the 100\(\tau\)% Bayesian prediction interval \((L, U)\) for \(X_{s,n}\) based on the Type-II hybrid censored sample is obtained by solving the following two equations:

\[
P(X_{s,n} > L | \bar{x}) = \frac{1 + \tau}{2},
\]

and

\[
P(X_{s,n} > U | \bar{x}) = \frac{1 - \tau}{2}. \quad (2.18)
\]
2.2. Two-sample Bayesian Prediction

Upon substituting (1.6) and (1.7) in (1.5), we obtain

\[ f(y_s \mid \mathbf{x}) = \sum_{w=0}^{s-1} C_3 \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)], \tag{2.19} \]

where \(1 \leq s \leq m\) and \(C_3 = \frac{(-1)^n (\tau^{-1})_m!}{(\tau^{-1})_m!}.\)

Since the Bayesian predictive density function of \(Y_{s,m}\) is given by

\[ p(y_s \mid \mathbf{x}) = \int_{\theta \in \Theta} f(y_s \mid \mathbf{x}) \pi^*(\theta \mid \mathbf{x}) d\theta, \tag{2.20} \]

upon substituting (2.10), (2.11), and (2.19) in (2.20), we obtain the Bayesian predictive density function of \(Y_{s,m}\) as follows:

**Case I.**

\[ p(y_s \mid \mathbf{x}) = \sum_{w=0}^{s-1} C_3 \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi^*_1(\theta \mid \mathbf{x}) d\theta; \tag{2.21} \]

**Case II.**

\[ p(y_s \mid \mathbf{x}) = \sum_{w=0}^{s-1} C_3 \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi^*_2(\theta \mid \mathbf{x}) d\theta. \tag{2.22} \]

From (2.21) and (2.22), we obtain the predictive survival function of \(Y_{s,m}\) as follows:

**Case I.**

\[ P(Y_{s,m} > t \mid \mathbf{x}) = \sum_{w=0}^{s-1} C_3 \int_{t}^{\infty} \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi^*_1(\theta \mid \mathbf{x}) d\theta d\lambda; \tag{2.23} \]

**Case II.**

\[ P(Y_{s,m} > t \mid \mathbf{x}) = \sum_{w=0}^{s-1} C_3 \int_{t}^{\infty} \int_{\theta \in \Theta} \lambda'(y_s, \theta) \exp[-(m-s+w+1)\lambda(y_s, \theta)] \pi^*_2(\theta \mid \mathbf{x}) d\theta d\lambda. \tag{2.24} \]

Consequently, the 100\(\pi\)% Bayesian prediction interval \((L, U)\) for \(Y_{s,m}\) based on the Type-II hybrid censored sample is obtained by solving the following two equations:

\[ P(Y_{s,m} > L \mid \mathbf{x}) = \frac{1 + \tau}{2}, \tag{2.25} \]
and

\[ P(Y_{x:n} > U \mid \mathbf{y}) = \frac{1 - \tau}{2}. \] (2.26)

3. Illustrative Examples

In this section, we discuss the Bayesian prediction problems for the exponential(\(\theta\)) distribution when \(\theta\) is unknown, and Pareto(\(x, \beta\)) distribution when both parameters \(x\) and \(\beta\) are unknown, as illustrative examples.

3.1. Exponential(\(\theta\)) Model

The distribution function in this case is

\[ F(x \mid \theta) = 1 - \exp(-\theta x), \quad x > 0, \] (3.1)

where \(\theta > 0\), and so we have

\[ \lambda(x; \theta) = \theta x \quad \text{and} \quad \lambda'(x; \theta) = \theta. \] (3.2)

For the case when \(\theta\) is unknown, we use the conjugate gamma prior for \(\theta\) with density

\[ \pi(\theta; \delta) = \frac{d^c}{\Gamma(c)} \theta^{c-1} \exp(-\theta d), \quad \theta > 0, \] (3.3)

where \(c\) and \(d\) are positive constants, and so we have

\[ C(\theta; \delta) = \theta^{\delta-1} \quad \text{and} \quad D(\theta; \delta) = \theta d, \] (3.4)

where \(\delta = (c, d)\).

3.1.1. One-sample Bayesian Prediction. The predictive survival function of \(X_{x:n}\) in this special case is as follows:

**Case I.**

\[
P(X_{x:n} > t \mid \mathbf{y}) = I_1^{-1} \sum_{w=0}^{r-1} \frac{C_1}{n - s + w + 1} \int_{0}^{\infty} \theta^{\delta-1} \exp \left[ -\theta \left( \sum_{i=1}^{r} x_i + (n - r) x_r + (n - s + w + 1)(t - x_r) + d \right) \right] d\theta
\]

\[ = I_1^{-1} \sum_{w=0}^{r-1} \frac{C_1}{n - s + w + 1} \times \left( \sum_{i=1}^{r} x_i + (n - r) x_r + (n - s + w + 1)(t - x_r) + d \right)^{-(\delta + 1)}, \] (3.5)
where

\[ I_1 = \left( \sum_{i=1}^{r} x_i + (n - r) \bar{x} + d \right)^{-(r+c)}; \]

**Case II.**

\[
P(Y_{x,m} > t | \mathbf{x}) = I_1^{-1} \sum_{w=0}^{s-1} \sum_{k=r}^{s-1} \frac{C_3}{m - s + w + 1} \int_0^{\infty} \phi_k(T, \theta) g_u(t, T; \theta) \pi_z^*(\theta | \mathbf{x}) d\theta, \quad (3.6)
\]

where

\[
\phi_k(T, \theta) = \binom{s}{k} \frac{\exp[-(n - k) \theta T](1 - \exp[-\theta T])^k}{\sum_{j=0}^{s-1} \binom{s}{j} \exp[-(n - j) \theta T](1 - \exp[-\theta T])^j},
\]

\[
g_u(t, T; \theta) = \exp[-\theta((n - s + w + 1)(t - T))],
\]

\[
\pi_z^*(\theta | \mathbf{x}) = \frac{d_1^{K+c}}{\Gamma(K + c)} \theta^{K+c-1} \exp(-\theta d_1),
\]

and \( d_1 = \sum_{i=1}^{K} x_i + (n - K) T + d \).

It does not seem to be possible to compute the probability in (3.6) analytically. We therefore use Markov Chain Monte Carlo (MCMC) technique for constructing the Bayesian prediction interval.

To compute \( \int_0^{\infty} f(\theta) \pi_z^*(\theta | \mathbf{x}) d\theta \) by using the MCMC technique, we use the following procedure:

**Step 1.** Generate \( \theta_1 \sim \text{Gamma}(K + c, \sum_{i=1}^{r} x_i + (n - K) T + d) \).

**Step 2.** Repeat Step 1 and obtain \( \theta_1, \theta_2, \ldots, \theta_N \).

**Step 3.** The approximate value of \( \int_0^{\infty} f(\theta) \pi_z^*(\theta | \mathbf{x}) d\theta \) is then obtained as

\[
\int_0^{\infty} f(\theta) \pi_z^*(\theta | \mathbf{x}) d\theta \approx \frac{\sum_{i=1}^{N} f(\theta_i)}{N}.
\]

3.1.2. **Two-sample Bayesian Prediction.** The predictive survival function of \( Y_{x,m} \) in this special case is as follows:

**Case I.**

\[
P(Y_{x,m} > t | \mathbf{x}) = I_1^{-1} \sum_{w=0}^{s-1} \frac{C_3}{m - s + w + 1} \int_0^{\infty} \theta^{K+c-1}
\]

\[
\times \exp \left[ -\theta \left( \sum_{i=1}^{r} x_i + (n - r) \bar{x} + (m - s + w + 1) t + d \right) \right] d\theta
\]

\[
= I_1^{-1} \sum_{w=0}^{s-1} \frac{C_3}{m - s + w + 1}
\]

\[
\times \left( \sum_{i=1}^{r} x_i + (n - r) \bar{x} + (m - s + w + 1) t + d \right)^{-(r+c)}, \quad (3.7)
\]
Bayesian Prediction based on Hybrid Censoring

where

\[ I_1 = \left( \sum_{j=1}^{r} x_j + (n-r)x_r + d \right)^{-(r+c)}; \]

**Case II.**

\[ P(Y_{x,m} > t \mid \mathbf{x}) = I_2^{-1} \sum_{w=0}^{r-1} \frac{C_3}{m-s+w+1} \int_0^\infty \theta^{K+c-1} \times \exp \left[ - \theta \left( \sum_{i=1}^{K} x_i + (n-K)T + (m-s+w+1)t + d \right) \right] d\theta \]

\[ = I_2^{-1} \sum_{w=0}^{r-1} \frac{C_3}{m-s+w+1} \times \left( \sum_{i=1}^{K} x_i + (n-K)T + (m-s+w+1)t + d \right)^{-(K+c)}, \quad (3.8) \]

where

\[ I_2 = \left( \sum_{i=1}^{K} x_i + (n-K)T + d \right)^{-(K+c)}. \]

**3.2. Pareto(\(\alpha, \beta\)) Model**

The distribution function in this case is

\[ F(x \mid \alpha, \beta) = 1 - \left( \frac{\beta}{x} \right)^\alpha \]

\[ = 1 - \exp \left[ - \alpha \ln \left( \frac{x}{\beta} \right) \right], \quad x > \beta, \quad (3.9) \]

where \(\alpha > 0\) and \(\beta > 0\), and so we have

\[ \lambda(x; \alpha, \beta) = \alpha \ln \left( \frac{x}{\beta} \right) \quad \text{and} \quad \lambda'(x; \alpha, \beta) = \frac{x}{\beta}. \]

Under the assumption that both parameters \(\alpha\) and \(\beta\) are unknown, we may consider a natural joint conjugate prior for \(\alpha\) and \(\beta\) which was suggested by Lwin (1972) and generalized by Arnold and Press (1989). The generalized Lwin prior or the power-gamma prior, denoted by PG\((a, b, c, d)\), is given by

\[ \pi(\alpha, \beta; d) \propto \alpha^a \beta^{b-1} \exp \left[ -\alpha \left( d + a \ln \left( \frac{b}{\beta} \right) \right) \right], \quad \alpha > 0, \quad 0 < \beta < b, \quad (3.11) \]

where \(a, b, c, d\) are positive constants. This general prior is obtained by first specifying the prior for the parameter \(\alpha\) and then specifying the conditional prior for \(\beta\), given knowledge on the parameter \(\alpha\). More specifically, we take \(\pi(\alpha)\) as a gamma
distribution with parameters $c$ and $d$, and $\pi(\beta \mid x)$ as a power function distribution with parameters $ax$ and $b$ of the form

$$
\pi(\beta \mid x) \propto x^b \beta^{ax-1} b^{-ax}, \quad 0 < \beta < b,
$$

to arrive at the joint prior given in (3.11). Thus, we have

$$
C(x, \beta; \delta) = x^\beta b^{-1} \quad \text{and} \quad D(x, \beta; \delta) = 2 \left( d + a \ln \left( \frac{b}{\beta} \right) \right), \quad (3.12)
$$

where $\delta = (a, b, c, d)$.

3.2.1. One-sample Bayesian Prediction. The predictive survival function of $X_{x,n}$ in this special case is as follows:

**Case I.**

$$
P(X_{x,n} > t \mid \mathbf{x}) = k^{-1} \sum_{w=0}^{s-r-1} \frac{C_1}{n - s + w + 1} \int_0^L \int_0^\infty z^{(r+c)} (r+c) b^{-l} \left( \prod_{i=1}^r \frac{1}{x_i} \right) \exp \left[ -z \left( \sum_{i=1}^r \ln \left( \frac{x_i}{\beta} \right) \right) \right] \, dz \, dB
$$

$$
+ (n - r) \ln \left( \frac{x_r}{\beta} \right) + a \ln \left( \frac{b}{\beta} \right) + (n - s + w + 1) \ln \left( \frac{t}{x_r} \right) + d = (n - s + w + 1) \ln \left( \frac{t}{x_r} \right) + d
$$

$$
l_{1} = \left( \sum_{i=1}^r \ln \left( \frac{x_i}{L} \right) + (n - r) \ln \left( \frac{x_r}{L} \right) + a \ln \left( \frac{b}{L} \right) + d \right)^{-(r+c)}, \quad (3.13)
$$

where $L = \min(x_1, b)$ and

$$
l_1 = \left( \sum_{i=1}^r \ln \left( \frac{x_i}{L} \right) + (n - r) \ln \left( \frac{x_r}{L} \right) + a \ln \left( \frac{b}{L} \right) + d \right)^{-(r+c)}.
$$

**Case II:**

$$
P(X_{x,n} > t \mid \mathbf{x}) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n - s + w + 1}
$$

$$
\times \int_0^L \int_0^\infty \phi_k(T; \alpha, \beta) g_w(t, T; \alpha, \beta) \pi_z^2(\alpha, \beta \mid \mathbf{x}) \, dz \, dB, \quad (3.14)
$$

where

$$
\phi_k(T; \alpha, \beta) = \frac{\left( \sum_{j=1}^k \right) \exp \left[ -(n - k) \alpha \ln \left( \frac{T}{\beta} \right) \right] \left( 1 - \exp \left[ -\alpha \ln \left( \frac{T}{\beta} \right) \right] \right)^k}{\sum_{j=1}^k \exp \left[ -(n - j) \alpha \ln \left( \frac{T}{\beta} \right) \right] \left( 1 - \exp \left[ -\alpha \ln \left( \frac{T}{\beta} \right) \right] \right)^j},
$$

$$
g_w(t, T; \alpha, \beta) = \exp \left[ -(n - s + w + 1) \alpha \ln \left( \frac{T}{T} \right) \right].
$$
Then use the MCMC technique for constructing the Bayesian prediction interval. We therefore use the Gibbs sampling technique to generate MCMC samples, and this special case is as follows:

\[ d_2 = \sum_{i=1}^{K} \ln \left( \frac{t_i}{\bar{S}} \right) + (n - K) \ln \left( \frac{t}{\bar{S}} \right) + a \ln \left( \frac{t + d}{\bar{S}} \right) + d. \]

It does not seem to be possible to compute the probability in (3.14) analytically. We therefore use the Gibbs sampling technique to generate MCMC samples, and then use the MCMC technique for constructing the Bayesian prediction interval.

To compute \( \int_0^L \int_0^\infty g(x, \beta) \pi^*_T(x, \beta | \underline{X}) \, dx \, d\beta \) by using the MCMC technique, we use the following procedure:

**Step 1.** Generate \( x_1 \sim \text{Gamma}(K + c, \sum_{i=1}^{K} \ln \left( \frac{x_i}{\bar{S}} \right) + (n - K) \ln \left( \frac{t}{\bar{S}} \right) + a \ln \left( \frac{t + d}{\bar{S}} \right) + d) \);

**Step 2.** Generate \( \beta_1 \sim \text{Power function}(x_1, (n + a), L);

**Step 3.** Repeat Steps 1 and 2 and obtain \( (x_1, \beta_1), (x_2, \beta_2), \ldots, (x_N, \beta_N) \);

**Step 4.** The approximate value of \( \int_0^L \int_0^\infty g(x, \beta) \pi^*_T(x, \beta | \underline{X}) \, dx \, d\beta \) is then obtained as

\[
\int_0^L \int_0^\infty g(x, \beta) \pi^*_T(x, \beta | \underline{X}) \, dx \, d\beta = \frac{\sum_{i=1}^{N} g(x_i, \beta_i)}{N}.
\]

### 3.2.2. Two-sample Bayesian Prediction

The predictive survival function of \( Y_{xm} \) in this special case is as follows:

**Case I.**

\[
P(Y_{xm} > t | \underline{X}) = I_1 \sum_{u=0}^{s-1} \sum_{m-s+w+1}^{C_3} \int_0^L \int_0^\infty x^{r+c} \left( \prod_{i=1}^{r} \frac{1}{x_i} \right) \exp \left[ -x \left( \sum_{i=1}^{r} \ln \left( \frac{x_i}{\bar{S}} \beta \right) \right) \right.
\]

\[
+ (n - r) \ln \left( \frac{x}{\bar{S}} \beta \right) + a \ln \frac{b}{\bar{S}} + (m - s + w + 1) \ln \left( \frac{t}{\bar{S}} \beta \right) + d \left] \, dx \, d\beta \right.
\]

\[
= I_1 \sum_{u=0}^{s-1} \frac{C_3}{(m - s + w + 1)(n + a + m - s + w + 1)}
\]

\[
\times \left( \sum_{i=1}^{r} \ln \left( \frac{x_i}{L} \right) + (n - r) \ln \left( \frac{x}{L} \right) + a \ln \frac{b}{L} \right.
\]

\[
+ (m - s + w + 1) \ln \left( \frac{t}{L} \right) + d \right)^{-(r+c)}, \quad (3.15)
\]

where

\[
I_1 = \frac{1}{n + a} \left( \sum_{i=1}^{r} \ln \left( \frac{x_i}{L} \right) + (n - r) \ln \left( \frac{x}{L} \right) + a \ln \frac{b}{L} + d \right)^{-(r+c)};
\]

**Case II.**

\[
P(Y_{xm} > t | \underline{X}) = I_2 \sum_{u=0}^{s-1} \sum_{m-s+w+1}^{C_3} \int_0^L \int_0^\infty x^{K+c} \left( \prod_{i=1}^{K} \frac{1}{x_i} \right) \exp \left[ -x \left( \sum_{i=1}^{K} \ln \left( \frac{x_i}{\bar{S}} \beta \right) \right) \right.
\]

\[
+ (n - K) \ln \left( \frac{t}{\bar{S}} \beta \right) + a \ln \frac{b}{\bar{S}} + (m - s + w + 1) \ln \left( \frac{L}{\bar{S}} \beta \right) + d \left] \, dx \, d\beta \right.
\]

\[
= I_2 \sum_{u=0}^{s-1} \frac{C_3}{(m - s + w + 1)(n + a + m - s + w + 1)}
\]

\[
\times \left( \sum_{i=1}^{K} \ln \left( \frac{x_i}{L} \right) + (n - K) \ln \left( \frac{x}{L} \right) + a \ln \frac{b}{L} \right.
\]

\[
+ (m - s + w + 1) \ln \left( \frac{t}{L} \right) + d \right)^{-(K+c)}, \quad (3.16)
\]

where

\[
I_2 = \frac{1}{n + a} \left( \sum_{i=1}^{K} \ln \left( \frac{x_i}{L} \right) + (n - K) \ln \left( \frac{x}{L} \right) + a \ln \frac{b}{L} + d \right)^{-(K+c)};
\]
\begin{align*}
&+ (n - K) \ln \left( \frac{T}{\beta} \right) + a \ln \frac{b}{\beta} + (m - s + w + 1) \ln \left( \frac{t}{\beta} \right) + d \right] dx \\
&= I_2^{-1} \sum_{w=0}^{r-1} \frac{C_j}{(m - s + w + 1)(n + a + m - s + w + 1)} \\
&\times \left( \sum_{i=1}^{K} \ln \left( \frac{x_i}{L} \right) + (n - K) \ln \left( \frac{T}{L} \right) + a \ln \frac{b}{L} \\
&+ (m - s + w + 1) \ln \left( \frac{t}{L} \right) + d \right)^{-1}.
\end{align*}

(3.16)

where

\[ I_2 = \frac{1}{n + a} \left( \sum_{i=1}^{K} \ln \left( \frac{x_i}{L} \right) + (n - K) \ln \left( \frac{T}{L} \right) + a \ln \frac{b}{L} + d \right)^{-1}. \]

4. Numerical Results

To illustrate the inferential procedures developed in the preceding sections, we present a numerical study for two distributions. The first distribution is the exponential(\(\theta\)) distribution when \(\theta\) is unknown, while the second one is the Pareto(\(\alpha, \beta\)) distribution when both parameters \(\alpha\) and \(\beta\) are unknown.

Example 4.1. To illustrate the prediction results for the exponential(\(\theta\)) distribution, when \(\theta\) is unknown, let us consider the data given by Bartholomew (1963) consisting of 20 items placed on a life test for a pre-fixed time of 150 h. During that period, 15 items failed with the following lifetimes, measured in hours: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, and 138.

We shall use these data to consider two different Type-II hybrid censoring schemes:

1. When \(r = 13\) and \(T = 100\). Since \(T > x_{13,20}\), the testing would have terminated in this case at \(T\) and we would have obtained the following data: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, and 99.
2. When \(r = 15\) and \(T = 100\). Since \(x_{15,20} > T\), the testing would have terminated in this case at time \(x_{15,20} = 138\) and we would have obtained the following data: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, and 138.

As done previously by Bartholomew (1963) and Childs et al. (2003), we assume these data to have come from the exponential(\(\theta\)) distribution, where \(\theta\) is unknown. Based on the above two Type-II hybrid censoring schemes, we then used the results presented earlier in sec. 3.1 to construct 95% one-sample Bayesian prediction intervals for future order statistics \(X_{s,m}, s = 16, \ldots, 20,\) from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics \(Y_{s,m}, s = 1, \ldots, 20,\) from a future sample of size \(m = 20\). To examine the sensitivity of the Bayesian prediction intervals with respect to the hyperparameters \((c, d)\), Table 1 presents the lower and upper 95% one-sample Bayesian prediction bounds for \(X_{k,m}, s = 16, \ldots, 20,\) for the choices of \(c = 0.9, 1, 1.1\) and \(d = 50, 55, 60\). The lower and
Table 1

95% one-sample Bayesian prediction bounds for $X_{s/n}$, $s = 16, \ldots, 20$, from the exponential distribution

\[
\begin{array}{ccccccc}
\text{c} & \text{s} & L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} \\
 & 17 & 103.193 & 283.859 & 103.231 & 284.627 & 103.270 & 285.394 \\
 & 18 & 107.335 & 357.982 & 107.437 & 359.038 & 107.540 & 360.094 \\
 & 19 & 117.996 & 475.301 & 118.232 & 476.802 & 118.469 & 478.303 \\
 & 20 & 145.245 & 739.427 & 145.670 & 741.923 & 146.096 & 744.419 \\
 & 17 & 103.125 & 282.131 & 103.164 & 282.895 & 103.202 & 283.657 \\
 & 19 & 117.604 & 471.901 & 117.836 & 473.391 & 118.070 & 474.879 \\
 & 20 & 144.538 & 733.824 & 144.961 & 736.301 & 145.384 & 738.776 \\
1.1 & 16 & 101.532 & 227.881 & 101.546 & 228.420 & 101.560 & 228.959 \\
 & 17 & 103.059 & 280.371 & 103.097 & 281.127 & 103.134 & 281.883 \\
 & 19 & 117.218 & 468.435 & 117.447 & 469.916 & 117.677 & 471.391 \\
 & 20 & 143.837 & 728.117 & 144.258 & 730.573 & 144.678 & 733.029 \\
\end{array}
\]

$r = 13$ and $T = 10$

\[
\begin{array}{ccccccc}
 & d = 50 & d = 55 & d = 60 & d = 50 & d = 55 & d = 60 \\
L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} \\
143.252 & 282.417 & 143.269 & 282.875 & 143.285 & 283.333 \\
169.317 & 483.830 & 169.417 & 484.926 & 169.516 & 486.023 \\
198.546 & 756.425 & 198.738 & 758.385 & 198.930 & 760.346 \\
\end{array}
\]

$r = 15$ and $T = 100$

\[
\begin{array}{ccccccc}
 & d = 50 & d = 55 & d = 60 & d = 50 & d = 55 & d = 60 \\
L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} & L_{X_{s/n}} & U_{X_{s/n}} \\
143.220 & 281.384 & 143.237 & 281.838 & 143.253 & 282.293 \\
169.131 & 481.304 & 169.230 & 482.392 & 169.329 & 483.480 \\
198.190 & 751.921 & 198.380 & 753.868 & 198.571 & 755.814 \\
\end{array}
\]
Table 2

95% two-sample Bayesian prediction bounds for $Y_{sn}$, $s = 1, \ldots, 20$, from the exponential distribution

<table>
<thead>
<tr>
<th>(c, d)</th>
<th>$r = 13$ and $T = 100$</th>
<th>$r = 15$ and $T = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1, 50)</td>
<td>(1, 55)</td>
</tr>
<tr>
<td>$s$</td>
<td>$L_{Y_{sn}}$</td>
<td>$U_{Y_{sn}}$</td>
</tr>
<tr>
<td>1</td>
<td>0.121</td>
<td>20.198</td>
</tr>
<tr>
<td>2</td>
<td>1.159</td>
<td>32.290</td>
</tr>
<tr>
<td>3</td>
<td>2.978</td>
<td>43.974</td>
</tr>
<tr>
<td>4</td>
<td>5.306</td>
<td>55.874</td>
</tr>
<tr>
<td>5</td>
<td>8.027</td>
<td>68.257</td>
</tr>
<tr>
<td>7</td>
<td>14.480</td>
<td>95.192</td>
</tr>
<tr>
<td>10</td>
<td>26.813</td>
<td>143.806</td>
</tr>
<tr>
<td>11</td>
<td>31.780</td>
<td>163.206</td>
</tr>
<tr>
<td>13</td>
<td>43.462</td>
<td>209.337</td>
</tr>
<tr>
<td>14</td>
<td>50.451</td>
<td>237.552</td>
</tr>
<tr>
<td>15</td>
<td>58.491</td>
<td>270.831</td>
</tr>
<tr>
<td>16</td>
<td>67.944</td>
<td>311.376</td>
</tr>
<tr>
<td>17</td>
<td>79.408</td>
<td>363.170</td>
</tr>
<tr>
<td>18</td>
<td>93.988</td>
<td>434.569</td>
</tr>
<tr>
<td>19</td>
<td>114.130</td>
<td>548.018</td>
</tr>
</tbody>
</table>
Bayesian Prediction based on Hybrid Censoring

upper 95% two-sample Bayesian prediction bounds for \( Y_{sm}, s = 1, \ldots, 20, \) for the choices of \( c = 1 \) and \( d = 50, 55, 60, \) are presented in Table 2.

**Example 4.2.** To illustrate the prediction results for the Pareto(\( \alpha, \beta \)) distribution, when both parameters \( \alpha \) and \( \beta \) are unknown, we generated order statistics from a sample of size \( n = 20 \) from the Pareto distribution. The generated order statistics from the Pareto distribution (with \( \alpha = 3 \) and \( \beta = 6 \)) are as follows: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, 9.290, 9.360, 9.525, 9.836, 10.263, 11.113, 15.769, and 39.211.

We shall use these data to consider two different Type-II hybrid censoring schemes.

1. When \( r = 13 \) and \( T = 9.3 \). Since \( T > x_{13,20} \), the testing would have terminated in this case at \( T \) and we would have obtained the following data: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, and 9.290.
2. When \( r = 15 \) and \( T = 9.3 \). Since \( x_{15,20} > T \), the testing would have terminated in this case at time \( x_{15,20} = 9.525 \) and we would have obtained the following data: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, 9.290, 9.360, and 9.525.

We assume these data to have come from the Pareto(\( \alpha, \beta \)) distribution, where both parameters \( \alpha \) and \( \beta \) are unknown. Based on the above two Type-II hybrid censoring schemes, we then used the results presented earlier in sec. 3.2 to construct 95% one-sample Bayesian prediction intervals for future order statistics \( X_{sm}, s = 16, \ldots, 20, \) from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics \( Y_{sm}, s = 1, \ldots, 20, \) from a future sample of size \( m = 20 \). To examine the sensitivity of the Bayesian prediction intervals with respect to the hyperparameters \( (a, b, c, d) \), we used three different choices of the hyperparameters \( (a, b, c, d) \): \((1, 9, 3, 1), (1, 9, 6, 2), (1, 9, 9, 3)\). The corresponding results for one-sample and two-sample predictions, for the three choices of the hyperparameters are presented in Tables 3 and 4, respectively.

**Remarks.**

1. From Tables 1–4, we notice that, when we use the same value of \( T \) but larger \( r \), the Bayesian prediction bounds become tighter as expected since the duration of the life-testing experiment is longer in this case.
2. It is evident from Tables 1 and 2 that, in the case of the exponential distribution, the lower as well as upper bounds are relatively insensitive to the specification of the hyperparameters \((c, d)\).
3. It is evident from Tables 3 and 4 that, in the case of the Pareto distribution, the lower bounds are relatively insensitive to the specification of the hyperparameters \((a, b, c, d)\) while the upper bounds are somewhat sensitive.
4. If the vector of prior parameters \( \delta \) is unknown, the empirical Bayes approach could be used in estimating such prior parameters based on past samples; see, for example, Maritz and Lwin (1989). Alternatively, one could use the hierarchical Bayesian method in which some suitable prior for \( \delta \) could be proposed; see, for example, Geisser (1990) and Bernardo and Smith (1994). Work in these directions are currently under progress and we hope to report these findings in a future article.
Table 3

95% two-sample Bayesian prediction bounds for $X_{s,n}$, $s = 16, \ldots, 20$, from the Pareto distribution

<table>
<thead>
<tr>
<th>$(a, b, c, d)$</th>
<th>$r = 13$ and $T = 9.3$</th>
<th>$r = 15$ and $T = 9.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$L_{X_{s,n}}$</td>
<td>$U_{X_{s,n}}$</td>
</tr>
</tbody>
</table>
Table 4
95% two-sample Bayesian prediction bounds for $Y_{s,m}$, $s = 1, \ldots, 20$, from the Pareto distribution

<table>
<thead>
<tr>
<th>$(a, b, c, d)$</th>
<th>$(1, 9, 3, 1)$</th>
<th>$(1, 9, 3, 2)$</th>
<th>$(1, 9, 4, 1)$</th>
<th>$(1, 9, 3, 1)$</th>
<th>$(1, 9, 3, 2)$</th>
<th>$(1, 9, 4, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>$L_{Y_{s,m}}$</td>
<td>$U_{Y_{s,m}}$</td>
<td>$L_{Y_{s,m}}$</td>
<td>$U_{Y_{s,m}}$</td>
<td>$L_{Y_{s,m}}$</td>
<td>$U_{Y_{s,m}}$</td>
</tr>
</tbody>
</table>
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References


