Applications of Derivatives

Optimization الأمثل

OR

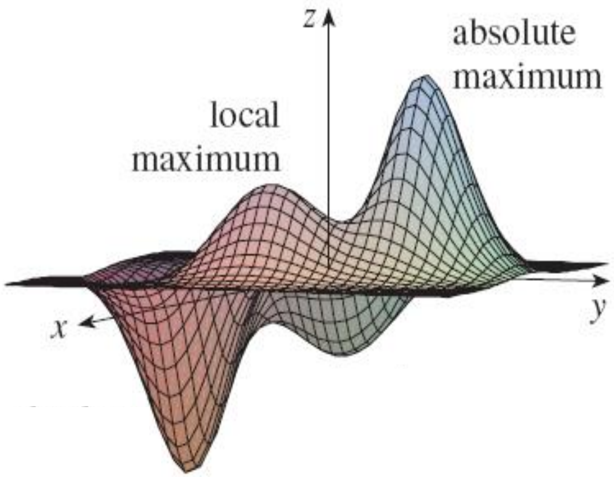
Maxima and Minima

Linear Algebra and Vector Analysis

We will learn how to use simple and partial derivatives to locate maxima and minima of functions of one and two variables.

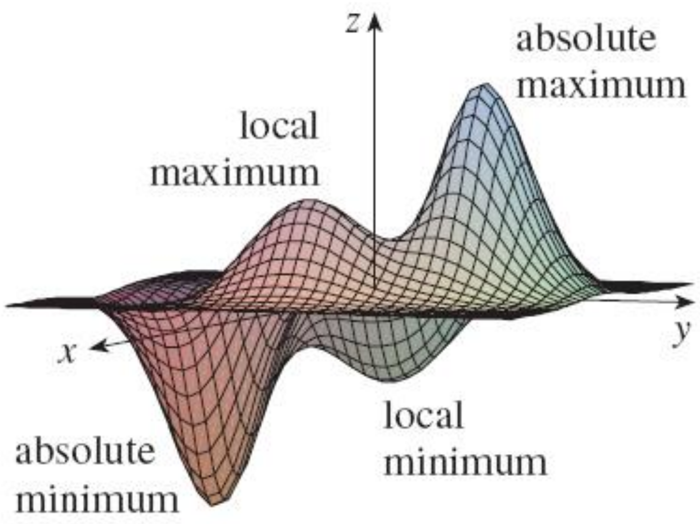
Absolute Maximum

- There are two points (a, b) where f has a local maximum—that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.
- The larger of these two values is the absolute maximum.



Absolute Minimum

- Likewise, *f* has two local minima—where $f(a, b)$ is smaller than nearby values.
- The smaller of these two values is the absolute minimum.



Example1: Find out maxima and minima of the following function

$$y = x^3 - 3x + 2$$

SOLUTION: For finding minima and maxima, first step is to find derivative then equate it to zero finding roots of the equation.

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

Now Equate the above equation to zero.

$$3(x^2 - 1) = 0$$

$$3(x-1)(x+1) = 0$$

$$Therefore$$

$$x = 1 \;\&\; x = -1$$

Put values of *x* equal to 1 and -1 into main function to get values of *y*

$$y = x^3 - 3x + 2$$

$$\text{when} \quad x = 1$$

$$y = (1)^3 - 3(1) + 2$$

$$y = 1 - 3 + 2 \Rightarrow 0$$

$$\text{when} \quad x = -1$$

$$y = (-1)^3 - 3(-1) + 2$$

$$y = -1 + 3 + 2 \Rightarrow 4$$

Now we have following two points which are called critical points.

(1,0) and (-1, 4)

Continue …

To evaluate critical points, we need second derivative of the function.

$$y = x^3 - 3x + 2$$

$$\frac{d^2y}{dx^2} = 6x$$

Name this second derivative "D". Put values of *x* from critical points into "D".

$$D = \frac{d^2y}{dx^2} = 6x$$

$$\text{when} \quad x = 1$$

$$D = 6$$

$$\text{when} \quad x = -1$$

$$D = -6$$

Conditions to check minima and maxima

- If D<0, the critical point is Maxima
- If D>0, the critical point is Minima

Therefore,

- At D = 6, (1,0) is a Minima
- At D = -6, (-1, 4) is Maxima

Example2: Find out maxima and minima of the following function

$$y = \frac{(x-1)^2}{x}$$

SOLUTION: Find first derivative of the function

$$y = \frac{(x-1)^2}{x}$$

$$\frac{dy}{dx} = \frac{x \times \frac{d}{dx}(x-1)^2 - (x-1)^2 \times \frac{d}{dx}x}{x^2}$$

$$\frac{dy}{dx} = \frac{x \times 2(x-1) - (x-1)^2 \times 1}{x^2}$$

$$\frac{dy}{dx} = \frac{2x(x-1) - (x-1)^2}{x^2}$$

Now Equate the above equation to zero.

$$\frac{2x(x-1) - (x-1)^2}{x^2} = 0$$

$$(x-1)(x+1) = 0$$

$$\textit{Therefore} \quad x = 1 \ \& \ x = -1$$

Put values of x equal to 1 and -1 into main function to get values of y

$$y = \frac{(x-1)^2}{x}$$

when $x = 1$

$$y = \frac{(1-1)^2}{1} \Rightarrow 0$$

when $x = -1$

$$y = \frac{(-1-1)^2}{-1} \Rightarrow -4$$

Now we have two critical points at (1,0) and (-1, -4)

Continue … To evaluate critical points, we need second derivative of the function.

$$y = \frac{(x-1)^2}{x}$$

$$\frac{dy}{dx} = \frac{(x-1)(x+1)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2 \frac{d}{dx}\left[(x-1)(x+1)\right] - \left[(x-1)(x+1)\frac{d}{dx}x^2\right]}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{x^2\left[(x-1)\times 1 + (x+1)\times 1\right] - (x-1)(x+1)\times 2x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{x^2(x-1) + x^2(x+1) - 2x(x-1)(x+1)}{x^4}$$

Name this second derivative "D". Put values of x from critical points into "D".

$$D = \frac{x^2(x-1) + x^2(x+1) - 2x(x-1)(x+1)}{x^4}$$

when $x = 1$

$D = 2$

when $x = -1$

$D = -2$

Conditions to check minima and maxima

- If D<0, the critical point is Maxima
- If D>0, the critical point is Minima

Therefore,

- At D = 2, (1,0) is a Minima
- At D = -2, (-1, -4) is Maxima

Example3: Find out maxima and minima of the following two variable function

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$$

SOLUTION: Partial differentiation is required for finding minima and maxima of two variable function.

$$f_x(x, y) = 6x^2 + 6y^2 - 150$$
$$f_y(x, y) = 12xy - 9y^2$$
$$f_{xx}(x, y) = 12x$$
$$f_{yy}(x, y) = 12x - 18y$$
$$f_{xy}(x, y) = 12y$$

For finding critical points we need to equate first derivative to zero and find roots of equations

$$6x^2 + 6y^2 - 150 = 0 \Rightarrow x^2 + y^2 = 25$$
$$12xy - 9y^2 = 0$$
$$y(12x - 9y) = 0 \quad \Rightarrow y = 0 \;\&\; x = \frac{3}{4}y$$

Put value of x into $x^2 + y^2 = 25$

After solving, we get

$$y = 4 \;\;\&\;\; y = \text{-}4 \; and \; x = 3 \;\&\; x = -3$$
$$Therefore \; (3,4), (-3,-4), (5,0), (-5,0)$$

For function of two variables;

$$D = f_{xx} f_{yy} - f_{xy}^{2}$$

$$When \quad (5,0)$$

$$D = (60)^2, \; f_{xx} = 60, \; f_{yy} = 60$$

$$When \quad (-5,0)$$

$$D = (60)^2, \; f_{xx} = -60, \; f_{yy} = -60$$

$$When \quad (3,4)$$

$$D = -3600$$

$$When \quad (-3,-4)$$

$$D = -3600$$

Conditions to check minima and maxima

- If D<0, at (a, b) then (a, b) is neither minima nor maxima i.e. Saddle Point.
- If D>0 and (f_{xx} <0 & f_{yy} <0) then (a, b) is a maximum point (maxima).
- If D>0 and (f_{xx} >0 & f_{yy} >0) then (a, b) is a minimum point (minima).

Therefore,

- (5,0) is a Minimum Point OR Minima.
- (-5,0) is a Maximum Point OR Maxima.
- (3,4) and (-3,-4) and Saddle Points.

Example 4: Find the maximum and minimum points of the following function.

$$\boxed{f(x,y) = x^4 + y^4 - 4xy + 1}$$

SOLUTION: Partial differentiation is required for finding minima and maxima of two variable function.

$$f(x,y) = x^4 + y^4 - 4xy + 1$$
$$f_x(x,y) = 4x^3 - 4y$$
$$f_{xx}(x,y) = 12x^2$$
$$f_y(x,y) = 4y^3 - 4x$$
$$f_{yy}(x,y) = 12y^2$$
$$f_{xy}(x,y) = 4$$

For finding critical points we need to equate first derivative with respect to "x" and "y" to zero and find roots of equations

$$x^3 - y = 0 \qquad (1)$$
$$y^3 - x = 0 \qquad (2)$$

From equation (1)

$\Rightarrow y = x^3$ Put in equation (2)

$$x^9 - x = 0$$
$$x(x^8 - 1) = 0$$
$$x(x^4 - 1)(x^4 + 1) = 0$$
$$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

- From the above equation values of x

$$x = 0, 1, -1$$

- Put these values of x into Equation (2) To get values of $y = 0,1,-1$
- Therefor three critical points are given $(0, 0), (1, 1), (-1, -1)$

$$D = f_{xx} f_{yy} - f_{xy}^2$$
$$When \quad (0,0)$$
$$D = -16$$
$$When \quad (1,1)$$
$$D = 128, \ f_{xx} = 12, \ f_{yy} = 12$$
$$When \quad (-1,-1)$$
$$D = 128, \ f_{xx} = 12, \ f_{yy} = 12$$

<u>Conditions to check minima and maxima</u>

- If D<0, at (a, b) then (a, b) is neither minima nor maxima i.e. Saddle Point.
- If D>0 and (f_{xx} <0 & f_{yy} <0) then (a, b) is a maximum point (maxima).
- If D>0 and (f_{xx} >0 & f_{yy} >0) then (a, b) is a minimum point (minima).

Therefore,

- (0,0) is a Saddle Point.
- (1, 1) & (-1, -1) are Minimum Point OR Minima.

Surface of the following function is given below. It does not have any maxima and it has two minima.

$$\boxed{f(x,y) = x^4 + y^4 - 4xy + 1}$$